

## Math 110, Summer 2012 Long Homework 4

Due Wednesday 8/1, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem we will study (the mathematical version of) the Casimir operator of the angular momentum algebra utilised in quantum mechanics. Casimir operators are used to represent angular momentum, elementary particle mass and spin and many other important quantities that arise in the study of elementary particles.

Let

$$\mathfrak{so}(3) = \{X \in \text{Mat}_3(\mathbb{R}) \mid X = -X^t\} \subset \text{Mat}_3(\mathbb{R}).$$

This is the *Lie algebra of the special orthogonal group*  $SO(3)$  - here

$$SO(3) = \{g \in O(3) \mid \det g = 1\}.$$

It is possible to identify  $SO(3)$  as the set of all rigid rotations of  $\mathbb{R}^3$  (ie, all possible rotations of the sphere). Essentially, elements of  $\mathfrak{so}(3)$  are the *infinitesimal generators* of rigid rotations.

a) Show that  $\mathfrak{so}(3)$  is a subspace of  $\text{Mat}_3(\mathbb{R})$  and that  $B = (L_x, L_y, L_z)$  is a basis of  $\mathfrak{so}(3)$ , where

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad L_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Deduce that  $\dim \mathfrak{so}(3) = 3$ .

b) Show that if  $X, Y \in \mathfrak{so}(3)$  then

$$\text{ad}(X)(Y) \stackrel{\text{def}}{=} XY - YX \in \mathfrak{so}(3).$$

Hence, we have defined a function

$$\text{ad}(X) : \mathfrak{so}(3) \rightarrow \mathfrak{so}(3); \quad Y \mapsto \text{ad}(X)(Y) = XY - YX.$$

Show that  $\text{ad}(X)$  is  $\mathbb{R}$ -linear, so that  $\text{ad}(X) \in \text{End}_{\mathbb{R}}(\mathfrak{so}(3))$ .

b) Determine the  $3 \times 3$  matrices

$$[\text{ad}(L_x)]_B, [\text{ad}(L_y)]_B, [\text{ad}(L_z)]_B.$$

c) Consider the bilinear form (you do NOT have to show  $B$  is a bilinear form)

$$B : \mathfrak{so}(3) \times \mathfrak{so}(3) \rightarrow \mathbb{R}; \quad (X, Y) \mapsto B(X, Y) = \text{tr}([\text{ad}(X)]_B [\text{ad}(Y)]_B).$$

i) Prove that, for every  $U, V \in \text{Mat}_3(\mathbb{R})$  we have  $\text{tr}(UV) = \text{tr}(VU)$ . Deduce that  $B$  is symmetric.

ii) Prove that  $B$  is nondegenerate.

iii) For  $L_x, L_y, L_z$  find  $M_x, M_y, M_z \in \mathfrak{so}(3)$  such that

$$B(L_i, M_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad \text{Here we take } i, j \in \{x, y, z\}.$$

d) Consider the matrix

$$\Omega = L_x M_x + L_y M_y + L_z M_z \in \text{Mat}_3(\mathbb{R}).$$

Show that  $\Omega$  is a scalar multiple of  $I_3$ .

$\Omega$  is called the *Casimir operator* and appears in the study of quantum angular momentum and leads to the definition of the quantum angular momentum number  $l$  (it is usually denoted  $L^2$  or  $J^2$  in QM textbooks). Our definition is slightly different than the one you'd see in QM, but only differs by a scalar multiple.

2. Consider the basis  $\mathcal{B} = (e, h, f) \subset \mathfrak{sl}_2(\mathbb{C})$ , defined in Long Homework 1. Here

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \text{Mat}_2(\mathbb{C}) \right\}.$$

Following the steps in Questions 1 (in particular, b), c), d); you DO NOT have to do a) or c)i) ) and using the basis  $\mathcal{B}$  show that the Casimir operator of  $\mathfrak{sl}_2(\mathbb{C})$  is

$$\Omega = \frac{1}{8}(h^2 + 2fe + 2ef) \in \text{Mat}_2(\mathbb{C}).$$

Deduce, that  $\Omega$  is a scalar multiple of  $I_2$ .