Math 110, Summer 2012 Long Homework 4

Due Wednesday 7/25, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem you will show that the nilpotent matrices with one Jordan block are the *regular nilpotent matrices* - this means that the nilpotent class of such matrices is the 'largest'.

Given $A \in Mat_n(\mathbb{C})$ we denote its similarity class by

$$\mathcal{O}(A) = \{B \in Mat_n(\mathbb{C}) \mid A \text{ is similar to } B\}.$$

a) Given $A \in Mat_n(\mathbb{C})$ we define the *commutator of A* to be

$$C(A) = \{B \in Mat_n(\mathbb{C}) \mid AB = BA\}.$$

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i) Show that $C(A) \subset Mat_n(\mathbb{C})$ is a subspace, for any $A \in Mat_n(\mathbb{C})$.

ii) Suppose that A and B are similar. **Fix** $P \in \mathrm{GL}_n(\mathbb{C})$ **such that** $P^{-1}AP = B$. Show that, for every <u>invertible</u> $X \in C(A)$, $Q^{-1}AQ = B$ where $Q = XP$.

iii) Let $Q \in Mat_n(\mathbb{C})$ be such that $Q^{-1}AQ = B$. Show that there is some <u>invertible</u> $Y \in C(A)$ such that $Q = YP$.

iv) Deduce that for every ordered basis $\mathcal{C} \subset \mathbb{C}^n$ such that $[T_A]_{\mathcal{C}} = B$ we can associate a <u>unique</u> invertible matrix $X(\mathcal{C}) \in C(A)$ such that $P_{\mathcal{S}\leftarrow\mathcal{C}} = X(\mathcal{C})P$. (*Hint: consider Corollary 1.7.7.*)

Therefore, we have a defined a function

$$\theta : \{\mathcal{C} \subset \mathbb{C}^n \mid [T_A]_{\mathcal{C}} = B\} \to C(A) \cap \mathrm{GL}_n(\mathbb{C})\ ;\ \mathcal{C} \mapsto X(\mathcal{C})$$

v) Show that θ is bijective. (*Hint: for surjectivity use Corollary 1.7.7.*)

We define the *dimension of $\mathcal{O}(A)$* to be $n^2 - \dim_{\mathbb{C}} C(A)$.[1]

b) Consider $Mat_3(\mathbb{C})$. There are three distinct nilpotent classes (as there are three partitions of 3) and any nilpotent $A \in Mat_3(\mathbb{C})$ is similar to precisely one of

$$N_{1^3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ N_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ N_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

i) Show that

$$C(N_{1^3}) = Mat_3(\mathbb{C}),\ C(N_{12}) = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mid d = f = g = 0, a = e \right\},$$

[1] The reason for this definition is (roughly) because we can consider

$$\mathcal{O}(A) = \{Q^{-1}AQ \mid Q \in \mathrm{GL}_n(\mathbb{C})\}.$$

Thus, we can define a surjective function

$$\alpha : \mathrm{GL}_n(\mathbb{C}) \to \mathcal{O}(A)\ ;\ Q \mapsto Q^{-1}AQ.$$

However, this function is not injective. In fact, for every $B \in \mathcal{O}(A)$ (say $P^{-1}AP = B$) we have

$$\alpha^{-1}(B) = \{Q \in \mathrm{GL}_n(\mathbb{C}) \mid \alpha(Q) = B\} = \{XP \mid X \in C(A)\}.$$

You have just shown that there is a bijection

$$\alpha^{-1}(B) \to C(A),$$

for any B. Thus, we could consider the measure of 'noninjectivity' to be $\dim_{\mathbb{C}} C(A)$. Then, we can consider the dimension of $\mathcal{O}(A)$ (= 'imα') to be $\dim GL_n(\mathbb{C}) - \dim C(A)$. This is a sort of geometric Rank Theorem result.

$$C(N_3) = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \mid d = g = h = 0, a = e = i, b = f \right\} = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in \mathbb{C} \right\}.$$

ii) Deduce that

$$\dim C(N_{1^3}) = 9, \ \dim C(N_{12}) = 5, \ \dim C(N_3) = 3,$$

and that $\mathcal{O}(N_3)$ has the largest dimension.

We will now (partially) show that the results we have obtained for the case $n = 3$ hold in general (ie, for every n we have $\dim \mathcal{O}(N_n)$ is maximal).

The following result will be useful: let $e_{ij} \in Mat_n(\mathbb{C})$ be the matrix with 0 everywhere except a 1 in the ij-entry. Then, we have

$$e_{ij}e_{kl} = \begin{cases} e_{il}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

You DO NOT have to show this.

c) Consider the nilpotent matrix N_n consisting of one 0-Jordan block. Thus, we have

$$N_n = e_{12} + e_{23} + \dots + e_{n-1,n} = \sum_{j=1}^{n-1} e_{j,j+1}.$$

i) Show that, for $1 \leq k, l \leq n$, we have

$$N_n e_{kl} - e_{kl} N_n = \begin{cases} -e_{1,l+1}, & \text{if } k = 1, \ 1 \leq l < n, \\ e_{k-1,n}, & \text{if } 1 < k \leq n, \ l = n, \\ e_{k-1,l} - e_{k,l+1}, & \text{if } k \neq 1, \ l \neq n, \\ 0, & \text{if } k = 1, l = n. \end{cases}.$$

ii) Show that, for each $i \in \{0, \dots, n-1\}$,

$$W_i \stackrel{def}{=} \text{span}_{\mathbb{C}}\{N_n e_{j,j+i} - e_{j,j+i} N_n \mid j+i \leq n \text{ and } j \geq 1\} = \text{span}_{\mathbb{C}}\{e_{j,j+i+1} \mid j+i+1 \leq n \text{ and } j \geq 1\}.$$

Deduce that $\dim W_i = n - 1 - i$.

You have just shown that the i^{th} diagonal[2] *of an arbitrary $n \times n$ matrix A is mapped* <u>*onto*</u> *the $(i+1)^{st}$ diagonal by the morphism $ad(N_n)$, for $i = 0, \dots, n-1$.*

iii) Show that, for each $i \in \{-1, \dots, -(n-1)\}$,

$$W_i = \text{span}_{\mathbb{C}}\{N_n e_{j+|i|,j} - e_{j+|i|,j} N_n \mid j+|i| \leq n, j \geq 1\} = \text{span}_{\mathbb{C}}\{e_{j+|i|-1,j} - e_{j+|i|,j+1} \mid j+|i|-1 \leq n \text{ and } j \geq 1\}.$$

Deduce that $\dim W_i = n + i$.

Hint: show that the set $\{e_{j+|i|-1,j} - e_{j+|i|,j+1} \mid j+|i| \leq n$ and $j \geq 1\}$ is linearly independent.

You have just shown that the i^{th} diagonal of an arbitrary $n \times n$ matrix A is mapped <u>*injectively into*</u> *the $(i+1)^{st}$ diagonal by the morphism $ad(N_n)$, for $i = -1, \dots, -(n-1)$.*

iv) Consider the morphism

$$ad(N_n) : Mat_n(\mathbb{C}) \to Mat_n(\mathbb{C}) \ ; \ B \mapsto N_n B - B N_n.$$

[2] We label the diagonals of an arbitrary $n \times n$ matrix as follows: the main diagonal is the 0^{th} diagonal and the diagonals to the right are labelled $1, \dots, n-1$ as move we move from left to right. The diagonals to the left of the main diagonal are labelled $-1, -2, \dots, -(n-1)$ as we move from right to left.

You have just determined the image of $ad(N_n)$ in ii)-iii): we have (you DO NOT need to justify this)

$$\text{im } ad(N_n) = W_{-(n-1)} \oplus W_{-(n-2)} \oplus \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{n-1}.$$

Deduce that $\dim ad(N_n) = n(n-1)$ and, using the Rank Theorem, deduce that

$$\dim C(N_n) = n.$$

(*Hint: what is* $\ker ad(N_n)$*?*)

In fact, it can be shown that

$$C(N_n) = \left\{ \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ 0 & c_1 & & \vdots \\ \vdots & & \ddots & c_2 \\ 0 & \cdots & & c_1 \end{bmatrix} \mid c_1, \dots, c_n \in \mathbb{C} \right\}.$$

d) Now, suppose that π is a partition of n such that $\pi \neq n$. Then, consider the block diagonal matrix

$$N_\pi = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix},$$

where each $J_i \in Mat_{n_i}(\mathbb{C})$ is a 0-Jordan block. (*So, we have* $\pi : n_1 + n_2 + \dots + n_k = n$*, where* $n_1 \geq n_2 \geq \dots \geq n_k > 0$.

Define, for each i,

$$m_i = n_1 + n_2 + \dots + n_i, \quad \text{and } m_0 = 0.$$

Show that

$$ad(N_\pi)(e_{m_i+1,m_j}) = 0, \quad \text{for each } i = 0, \dots, k-1 \text{ and } j = 1, \dots, k,$$

and deduce that $\dim C(N_\pi) \geq k^2$. In particular, if $k^2 \geq n$ then

$$\dim C(N_\pi) \geq \dim C(N_n),$$

and

$$\dim \mathcal{O}(N_n) \geq \dim \mathcal{O}(N_\pi).$$