Math 110, Summer 2012 Long Homework 3

Due Tuesday 7/10, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. In this problem you will prove that commuting diagonalisable matrices can be *simultaneously diagonalised.*[1]

Let $f, g \in \mathrm{End}_{\mathbb{C}}(V)$, where V is a finite dimensional $\mathbb{C}$-vector space.

a) Suppose that U is a g-invariant subspace of V. Let $E^g_{\mu_i}$ be the μ_i-eigenspace of g (so that μ_i is an eigenvalue of g). Show that

$$\left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U = (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U),$$

as follows:

i) Show that
$$E^g_{\mu_1} \cap U + ... + E^g_{\mu_k} \cap U \subset \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U.$$

ii) If $W_i = E^g_{\mu_i} \cap U$ show that
$$W_i \cap (\sum_{j \neq i} W_j) = \{0_V\}, \quad \text{for each } i.$$

Hence, we have
$$E^g_{\mu_1} \cap U + ... + E^g_{\mu_k} \cap U = (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

Suppose that $u \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U$. Then, $u \in U$ and

$$u = e_1 + ... + e_k,$$

with $e_i \in E^g_{\mu_i}$. You are now going to show that $e_i \in U$, for each i, thereby showing that

$$u \in (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

Let
$$\Gamma_1 = \{i \in \{1, ..., k\} \mid e_i \in U\}, \ \Gamma_2 = \{i \in \{1, ..., k\} \mid e_i \notin U\},$$

so that $\Gamma_1 \cup \Gamma_2 = \{1, ..., k\}$.

iii) Show that if $\Gamma_2 = \varnothing$ then $u \in (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U)$.

iv) Show that if $\Gamma_2 \neq \varnothing$ then
$$u - \sum_{j \in \Gamma_1} e_j \in U.$$

Deduce that if $\Gamma_2 \neq \varnothing$ then there is some nonzero $w \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U$, such that

$$w = e_{i_1} + ... + e_{i_s}, \quad \text{with } e_{i_j} \in E^g_{\mu_{i_j}} \text{ and } e_{i_j} \notin U.$$

[1] In fact, if we have a family (A_i) of diagonalisable matrices, such that $A_i A_j = A_j A_i$, for every i, j, then there is a common eigenbasis of all of the A_i: this means there is a *single* matrix P such that

$$P^{-1} A_i P = D_i,$$

with D_i diagonal, for every i.

v) Suppose $\Gamma_2 \neq \varnothing$ and let

$$\mathcal{L} = \{w \in \left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U \mid w = e_{i_1} + \ldots + e_{i_s}, \text{ with } e_{i_j} \in E^g_{\mu_{i_j}} \text{ and } e_{i_j} \notin U\}.$$

By *iv*) we know that $\mathcal{L} \neq \varnothing$. Let $w \in \mathcal{L}$ with

$$w = e_{i_1} + \ldots + e_{i_s}.$$

Show that it is not possible for $s = 1$. Deduce that we must have $s \geq 2$.

vi) Let $w \in \mathcal{L}$ with

$$w = e_{i_1} + \ldots + e_{i_s},$$

and such that s is minimal. Using v) deduce that there is some $j \in \{1, \ldots, s\}$ such that e_{i_j} is an eigenvector associated to a <u>nonzero</u> eigenvalue μ_{i_j} and show that

$$g(w) - \mu_{i_j} w \in \mathcal{L}.$$

Explain why we have contradicted the minimality condition for w.

vii) Explain why $\Gamma_2 = \varnothing$ and deduce that

$$\left(E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k}\right) \cap U \subset (E^g_{\mu_1} \cap U) \oplus \cdots \oplus (E^g_{\mu_k} \cap U).$$

b) Deduce that if g admits a basis of eigenvectors then there is a basis of U (we are still assuming that U is g-invariant) consisting of eigenvectors of g. (*Hint: Use that* $E^g_{\mu_1} \oplus \cdots \oplus E^g_{\mu_k} = V$ *and* a).)

c) Suppose that $f \circ g = g \circ f$ (we say that f and g *commute*). Let E^f_λ be the λ-eigenspace of f. Prove that E^f_λ is g-invariant. (*Hint: You must show that if* $v \in E^f_\lambda$ *then* $g(v) \in E^f_\lambda$.)

d) Deduce that if f and g commute and there exists a basis of V consisting of eigenvectors of g then there exists a basis of $E^f_{\lambda_i}$ consisting of eigenvectors of g, for every eigenvalue λ_i of f. (*Hint: Use* b) *and* c).)

e) Prove: if f and g commute and there exists two bases of V, one consisting of eigenvectors of f and the other consisting of eigenvectors of g, then there is a single basis of V consisting of eigenvectors of <u>both f and g</u>.

f) Prove: Let $A, B \in Mat_n(\mathbb{C})$ such that $AB = BA$. Suppose that A and B are both diagonalisable. Then, there is an invertible matrix P such that

$$P^{-1}AP = D_1, \quad P^{-1}BP = D_2,$$

with D_i a diagonal matrix. (*Hint: This follows from* e).)

g) Find an invertible matrix P such that

$$P^{-1}AP = D_1, \quad P^{-1}BP = D_2,$$

with D_i diagonal, and where

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & -4 \\ 0 & 3 \end{bmatrix}.$$

(*You do not need to show that* $AB = BA$ *or that* A *and* B *are diagonalisable - although you should be able to see that they are diagonalisable by looking at them.*)