Math 110, Summer 2012 Long Homework 2

Due Tuesday 7/3, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. Let V be a $\mathbb{K}$-vector space. Consider the following *minimal spanning* property of a spanning set $E \subset V$ (so that $\text{span}_{\mathbb{K}} E = V$):

- $*$ if $E' \subset E$ and $\text{span}_{\mathbb{K}} E = V$ then $E' = E$.

Prove that a subset $E \subset V$ that spans V, so that $\text{span}_{\mathbb{K}} E = V$, and which satisfies the minimal spanning property $*$ is a basis of V.

(*This is Proposition 1.5.9 on p.32 of the notes. To show that E is a basis of V it suffices to show that E is linearly independent. Look at the proof of Proposition 1.5.5 to help you show how you can use the minimal spanning property of E to obtain linear independence of E.*)

2. In this problem we are going to try and determine properties of a matrix $A \in Mat_n(\mathbb{K})$ by studying endomorphisms of $Mat_n(\mathbb{K})$.

Let $A \in Mat_n(\mathbb{K})$. Define the linear morphisms (you DO NOT have to show this)

$$L_A : Mat_n(\mathbb{K}) \to Mat_n(\mathbb{K})\ ;\ B \mapsto AB, \quad R_A : Mat_n(\mathbb{K}) \to Mat_n(\mathbb{K})\ ;\ B \mapsto BA.$$

a) Prove that L_A is injective if and only if A is invertible.

b) Prove that R_A is injective if and only if A is invertible.

c) Deduce that L_A is injective if and only if R_A is injective.

(*Theorem 1.7.4 might be useful for parts a), b).*)

Now, consider

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in Mat_2(\mathbb{Q}),$$

and let $\mathcal{B} = (e_{11}, e_{12}, e_{21}, e_{22})$ denote the standard ordered basis of $Mat_2(\mathbb{Q})(= \mathbb{K}^{[2]\times[2]})$.

We know, using determinants for example, that A is invertible. However, we are going to obtain this fact using the results you have just proved above.

d) Determine the matrix of L_A relative to $\mathcal{B}$, $[L_A]_{\mathcal{B}} \in Mat_4(\mathbb{Q})$.

e) Show that L_A is injective. (*Use Theorem 1.7.4*)

f) Deduce that A is invertible.

g) By solving the matrix equation

$$[L_A]_{\mathcal{B}} \underline{x} = [I_2]_{\mathcal{B}},$$

find the inverse of A.