Math 110, Summer 2012 Long Homework 1

Due Tuesday 6/26, 10.10am, in Etcheverry 3109. Late homework will not be accepted.

Please write your answers in complete English sentences (where applicable). Make your arguments rigorous - if something is 'obvious', state why this is the case. Full credit will be awarded to those solutions that are complete and answer the question posed in a coherent manner.

1. Let V be a $\mathbb{K}$-vector space, $E \subset V$ a nonempty finite subset. In this question we will characterise properties of E using linear morphisms.

a) Prove: E is linearly independent if and only if the linear morphism

$$h : \mathbb{K}^E \to V \; ; \; f \mapsto \sum_{e \in E} f(e) \cdot e,$$

is injective.

b) Prove: E is a spanning set of V if and only if the linear morphism

$$h : \mathbb{K}^E \to V \; ; \; f \mapsto \sum_{e \in E} f(e) \cdot e,$$

is surjective.

2. Consider the subspace

$$sl_2(\mathbb{C}) = \left\{ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in Mat_2(\mathbb{C}) \mid a_{11} + a_{22} = 0 \right\} \subset Mat_2(\mathbb{C}).$$

So, $sl_2(\mathbb{C}) = \ker \mathrm{tr}$, where tr is the linear morphism (you DO NOT have to show this)

$$\mathrm{tr} : Mat_2(\mathbb{C}) \to \mathbb{C} \; ; \; A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto a_{11} + a_{22}.$$

This is the *special linear Lie (pronounced 'lee') algebra of* 2×2 *complex matrices*. It is of fundamental importance and arises in many areas of mathematics. We denote

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in sl_2(\mathbb{C}).$$

a) Using the Rank Theorem, show that $\dim sl_2(\mathbb{C}) = 3$.

b) Show that $\mathcal{B} = (e, h, f)$ is an ordered basis of $sl_2(\mathbb{C})$.

For every $A \in M_2(\mathbb{C})$ we have a function

$$\mathrm{ad}_A : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \; ; \; B \mapsto AB - BA.$$

c) Show that ad_A is a linear morphism, for every $A \in M_2(\mathbb{C})$.

d) Let $A, B \in sl_2(\mathbb{C})$. Show that $\mathrm{ad}_A(B) \in sl_2(\mathbb{C})$.

Hence, for $A \in sl_2(\mathbb{C})$ we see that $\mathrm{ad}_A \in \mathrm{End}_{\mathbb{C}}(sl_2(\mathbb{C}))$ so that there exists a function

$$\mathrm{ad} : sl_2(\mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(sl_2(\mathbb{C})) \; ; \; A \mapsto \mathrm{ad}_A.$$

e) Determine $[\mathrm{ad}_e]_{\mathcal{B}}, [\mathrm{ad}_h]_{\mathcal{B}}, [\mathrm{ad}_f]_{\mathcal{B}}$, the matrices of $\mathrm{ad}_e, \mathrm{ad}_h, \mathrm{ad}_f$ with respect to the ordered basis $\mathcal{B}$.

f) The function ad is linear (you DO NOT have to show this): hence, if $A = \lambda e + \mu h + \tau f \in sl_2(\mathbb{C}), \lambda, \mu, \tau \in \mathbb{C}$, then $\mathrm{ad}_A = \lambda \mathrm{ad}_e + \mu \mathrm{ad}_h + \tau \mathrm{ad}_f$. Show that ad is injective.