Middlebury College

Multivariable Calculus
Spring 2018
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MAY 9 LECTURE

TEXTBOOK REFERENCE:
- *Vector Calculus*, Colley, 4th Edition: §5.2, 5.3

DOUBLE INTEGRALS II

LEARNING OBJECTIVES:
- *Learn what an elementary region is.*
- *Learn how to compute double integrals defined on elementary regions.*
- *Learn how to change the order of integration.*

KEYWORDS: elementary regions, changing the order of integration

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Integrating over general bounded regions

Suppose that D is a region of the form

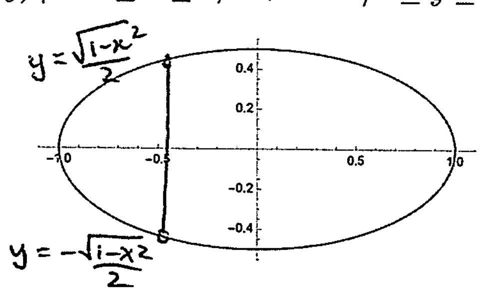
$$D = \{(x,y) \mid a \leq x \leq b,\ c(x) \leq y \leq d(x)\}$$

We call such regions D **elementary regions of Type 1**. For example, the interior of an ellipse

$$D = \{(x,y) \mid x^2 + 4y^2 \leq 1\}$$

is such a region:

$$D = \{(x,y) \mid -1 \leq x \leq 1,\ -\sqrt{1-x^2}/2 \leq y \leq \sqrt{1-x^2}/2\}$$



> The above description of D is obtained by considering D as a collection of vertical line segments:
>
> *For every $x \in [-1,1]$ the vertical line segment lying between $-\sqrt{1-x^2}/2$ and $\sqrt{1-x^2}/2$ lies within D. Moreover, D is the collection of all such line segments.*

If $f(x,y)$ is continuous on an elementary region D of Type 1 then we define

$$\int\int_D f dA = \int_{x=a}^{x=b} \int_{y=c(x)}^{y=d(x)} f(x,y)dydx$$

This double integral computes the (signed) volume of the region lying between the graph $z = f(x,y)$ and D.

Remark:

Example: Let $f(x,y) = x$, D be the interior of the ellipse above. Then,

$$\int\int_R f dA = \int_{x=-1}^{x=1} \left(\int_{y=-\sqrt{1-x^2}/2}^{y=\sqrt{1-x^2}/2} x dy \right) dx$$
$$= \int_{x=-1}^{x=1} \left([xy]_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} \right) dx$$
$$= \int_{x=-1}^{x=1} \left(x\sqrt{1-x^2} \right) dx$$
$$= \left[-\frac{1}{3}(1-x^2)^{3/2} \right]_{-1}^{1} = 0$$

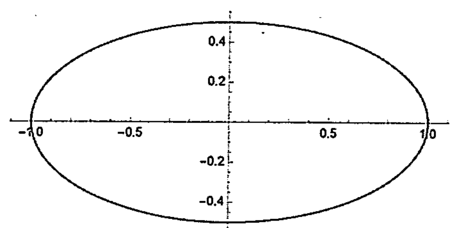
Remark: since the region D is symmetric in the y-axis and $f(x,y) = -f(-x,y)$, we could use symmetry to deduce the answer.

Important Observation: The region D given above could also be described as

$$D = \{(x,y) \mid -1/2 \leq y \leq 1/2, \ -\sqrt{1-4y^2} \leq x \leq \sqrt{1-4y^2}\}$$

The above description of D is obtained by considering D as a collection of horizontal line segments:

For every $y \in [-1/2, 1/2]$ the horizontal line segment lying between $-\sqrt{1-4y^2}$ and $\sqrt{1-4y^2}$ lies within D. Moreover, D is the collection of all such line segments.



Remark:

1. If D is a region in the plane such that

$$D = \{(x,y) \mid c \leq y \leq d, \ a(y) \leq x \leq b(y)\}$$

then we call D an **elementary region of Type 2.** If D is an elementary region of type 2 and $f(x,y)$ is continuous on D then we define

$$\int\int_D f dA = \int_{y=c}^{y=d} \int_{x=a(y)}^{x=b(y)} f(x,y) dx dy$$

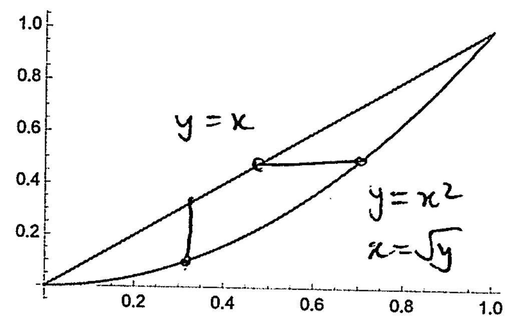
2. A region D that is both a Type 1 elementary region and a Type 2 elementary region is called an **elementary region of Type 3.**

Example:

1. The ellipse defined above is an elementary region of Type 3. Therefore, we can also compute the double integral of $f(x,y) = x$ over D as follows:

$$\int\int_R f dA = \int_{y=-1/2}^{y=1/2} \left(\int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} x dx \right) dy$$
$$= \int_{y=-1/2}^{y=1/2} \left(\left[\frac{x^2}{2} \right]_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \right) dy$$
$$= \int_{y=-1/2}^{y=1/2} 0 dy = 0$$

2. Let D be the region lying between the graph $y = x^2$ and the line $y = x$.



D is an elementary region of type 3:

Type 1:

$0 \leq x \leq 1$

$x^2 \leq y \leq x$

Type 2:

$0 \leq y \leq 1$

$y \leq x \leq \sqrt{y}$

Changing the order of integration

Given an elementary region D of type 3 and a continuous function on D, we can compute the double integral $\int\int_D f dA$ using two different iterated integrals. Sometimes, it can be useful to **change the order of integration** to determine the double integral.

Example: We want to compute the iterated integral:

$$\int_0^2 \int_{y^2}^4 y \cos(x^2) dx dy$$

However, we immediately run into trouble: **it's impossible to find an antiderivative of $\cos(x^2)$ that can be expressed using well-known functions.** However, all is not lost.

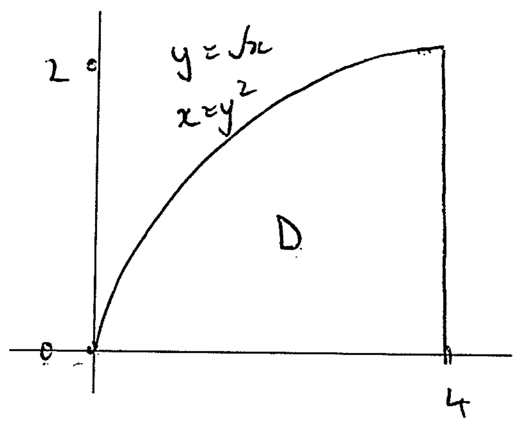
This iterated integral is equal to

$$\int\int_D y\cos(x^2)dA$$

for some region D. Observing the order of $dxdy$ in the integral tells us that we **integrate with respect to x first and then with respect to y.** Thus, the integrations bounds are describing a Type 2 region:

$$D = \{(x, y) \mid 0 \leq y \leq 2,\ y^2 \leq x \leq 4\}$$

Diagram:



Therefore, D is also a type 1 region

$$D = \{(x, y) \mid 0 \leq x \leq 4,\ 0 \leq y \leq \sqrt{x}$$

Hence

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)dxdy = \int\int_D y\cos(x^2)dA$$

$$= \int_{x=0}^{4} \int_{y=0}^{y=\sqrt{x}} y\cos(x^2)\,dy\,dx$$

$$= \int_{x=0}^{x=4} \left[\frac{x}{2}\cos(x^2)\right]dx$$

$$= \left[\frac{1}{4}\sin(x^2)\right]_0^4$$

$$= \frac{1}{4}\sin(16).$$