



## MAY 7 LECTURE

### TEXTBOOK REFERENCE:

- *Vector Calculus*, Colley, 4th Edition: §5.2

## DOUBLE INTEGRALS

### LEARNING OBJECTIVES:

- Learn what a double integral is.
- Learn how to compute basic double integrals.

KEYWORDS: double integrals

### Integrating over rectangles

Let  $f(x, y)$  be a function defined on the rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

Choose partitions of  $[a, b]$  and  $[c, d]$  into  $n$  subintervals

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b, \quad c = y_0 < y_1 < \dots < y_{n-1} < y_n = d$$

Denote  $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ , and write

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}, \quad \Delta_{ij} = \Delta x_i \Delta y_j.$$

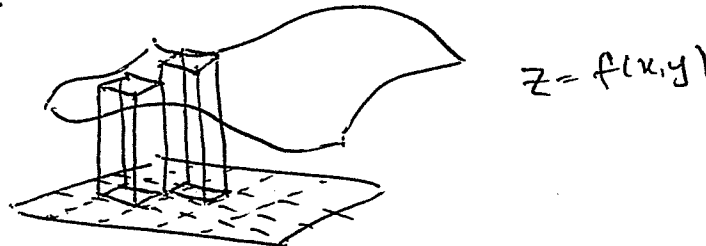
For  $1 \leq i, j \leq n$ , choose  $c_{ij} \in R_{ij}$ . The quantity

$$\sum_{i,j=1}^n f(c_{ij}) \Delta_{ij}$$

↖ base  
↑ weight

is called a Riemann sum of  $f$  on  $R$  corresponding to the partition.

Diagram:



A Riemann sum of  $f$  on a rectangle  $R$  provides an approximation to the (signed) volume of the region lying between the graph  $z = f(x, y)$  and the  $xy$ -plane.

### Double Integral:

The double integral of  $f$  on  $R$ , denoted  $\int \int_R f dA$  is the limit of the Riemann sums  $S$  obtained by letting  $\Delta x_i, \Delta y_j \rightarrow 0$

$$\int \int_R f dA = \lim_{\Delta x_i, \Delta y_j \rightarrow 0} \sum_{i,j=1}^n f(c_{ij}) \Delta_{ij}$$

should this limit exist. When the limit exists we say that  $f$  is integrable on  $R$ .

#### Remark:

1. If  $f$  is integrable on  $R$  then  $\int \int_R f dA$  computes the (signed) volume of the region lying between the graph  $z = f(x, y)$  and the region  $R$ .
2. It is a fact that, if  $f$  is integrable on  $R$  then  $\int \int_R f dA$  can be computed using any partition of  $R$ .

As in the single variable case, we can assure the existence of the limit by assuming that  $f$  is reasonably nice.

### Continuous $\implies$ integrable:

Let  $f(x, y)$  be a continuous function defined on a (bounded) rectangle  $R$ . Then,  $f$  is integrable on  $R$ .

More generally, if  $f(x, y)$  is a, not necessarily continuous, bounded function (i.e. no infinite discontinuities) and the set of discontinuities of  $f$  has zero area, then  $f$  is integrable on  $R$ .

**Remark:** The above result does not provide us with an approach to **computing double integrals**: to determine  $\int \int_R f dA$ , for continuous  $f$ , we would have to form Riemann sums and take limits (which may not be so easy to do!).

**Example:** Consider the function  $f(x, y) = 2x + y$ . Then, the graph  $z = 2x + y$  is a plane. Let  $R = [0, 1] \times [0, 1]$  and consider the partition

$$0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < 1$$

so that  $x_i = \frac{i}{n}$ ,  $y_j = \frac{j}{n}$ . Let  $c_{ij} = (\frac{i}{n}, \frac{j}{n}) \in R_{ij} = [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ . Then, the corresponding Riemann sum is

$$S = \sum_{i,j=1}^n f(c_{ij}) \Delta_{ij} = \frac{\sum_{i,j} (2\frac{i}{n} + \frac{j}{n}) \frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{n^3} \sum_{i,j=1}^n (2i + j)$$

Since

$$\sum_{i,j=1}^n (2i + j) = \sum_{i=1}^n \left( \sum_{j=1}^n (2i + j) \right)$$

we find

$$S = \frac{1}{n^3} \sum_{i=1}^n \left( 2in + \frac{1}{2}n(n+1) \right) = \frac{1}{n^3} \left( 2n \cdot \frac{1}{2}n(n+1) + \frac{1}{2}n^2(n+1) \right) = \frac{3}{2} \left( 1 + \frac{1}{n} \right)$$

$$\implies \iint_R f dA = \lim_{n \rightarrow \infty} \frac{3}{2} \left( 1 + \frac{1}{n} \right) = \frac{3}{2}$$

Here we use the identities

$$\sum_{i=1}^n i = \frac{1}{2}n(n+1), \quad \sum_{i=1}^n k = kn, \quad \text{for any constant } k$$

### Fubini's Theorem

The relationship between double integration and (partial) differentiation is not as straightforward as the single variable case. In particular, trying to compute double integrals without using Riemann sums can be tricky. However, in the case considered above we have the following result

#### Fubini's Theorem:

Let  $f(x, y)$  be a continuous function defined on a (bounded) rectangle  $R = [a, b] \times [c, d]$ . Then,

$$\iint_R f dA = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y) dy \right) dx = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

The integrals appearing the right hand side are called **iterated integrals**.

**Remark:** We will usually write

$$\int_a^b \int_c^d f(x, y) dy dx = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y) dy \right) dx$$

and

$$\int_c^d \int_a^b f(x, y) dx dy = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y) dx \right) dy$$

In particular, pay attention to the order of  $dx dy$  or  $dy dx$ .

**Example:**

1. Let  $f(x, y) = 2x + y$ ,  $R = [0, 1] \times [0, 1]$ . Then, using Fubini's Theorem we can compute

$$\begin{aligned} \iint_R f dA &= \int_{x=0}^{x=1} \int_{y=0}^{y=1} (2x + y) dy dx \\ &= \int_{x=0}^{x=1} \left[ 2xy + \frac{y^2}{2} \right]_0^1 dx \\ &= \int_{x=0}^{x=1} \left( 2x + \frac{1}{2} \right) dx \\ &= \left[ x^2 + x/2 \right]_0^1 = 3/2 \end{aligned}$$

You can check that we get the same answer if we compute the integral with respect to  $x$  first.

2. Let  $f(x, y) = \cos(x) \cos(y)$ ,  $R = [0, \pi] \times [\pi/4, \pi/2]$ . Then, by Fubini's Theorem

$$\begin{aligned} \iint_R f dA &= \int_{x=0}^{x=\pi} \int_{y=\pi/4}^{y=\pi/2} \cos(x) \cos(y) dy dx \\ &= \int_{x=0}^{x=\pi} [\cos(x) \sin(y)]_{\pi/4}^{\pi/2} dx \\ &= \int_{x=0}^{x=\pi} (\cos(x)(1 - 1/\sqrt{2})) dx \\ &= [\sin(x)(1 - 1/\sqrt{2})]_0^\pi = 0 \end{aligned}$$

**Remark:** See p.320 of the textbook for the basic properties of a double integral.

### Integrating over general bounded regions

Suppose that  $D$  is a region of the form

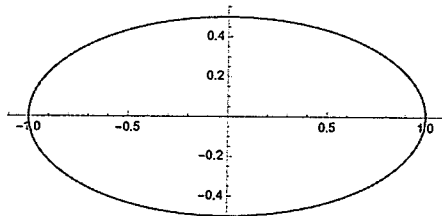
$$D = \{(x, y) \mid a \leq x \leq b, c(x) \leq y \leq d(x)\}$$

We call such regions  $D$  **elementary regions of Type 1**. For example, the interior of an ellipse

$$D = \{(x, y) \mid x^2 + 4y^2 \leq 1\}$$

is such a region:

$$D = \{(x, y) \mid -1 \leq x \leq 1, -\sqrt{1-x^2}/2 \leq y \leq \sqrt{1-x^2}/2\}$$



If  $f(x, y)$  is continuous on an elementary region  $D$  of Type 1 then we define

$$\iint_D f dA = \int_{x=a}^{x=b} \int_{y=c(x)}^{y=d(x)} f(x, y) dy dx$$

This double integral computes the (signed) volume of the region lying between the graph  $z = f(x, y)$  and  $D$ .

**Example:** Let  $f(x, y) = x$ ,  $D$  be the interior of the ellipse above. Then,

$$\begin{aligned} \iint_R f dA &= \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}/2}^{y=\sqrt{1-x^2}/2} x dy dx \\ &= \int_{x=-1}^{x=1} [xy]_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} dx \\ &= \int_{x=-1}^{x=1} (x\sqrt{1-x^2}) dx \\ &= \left[ -\frac{1}{3}(1-x^2)^{3/2} \right]_{-1}^1 = 0 \end{aligned}$$

Observe: since the region  $D$  is symmetric in the  $y$ -axis and  $f(x, y) = -f(-x, y)$ , we could use symmetry to deduce the answer.