Middlebury College

Multivariable Calculus
Spring 2018
Contact: `gmelvin@middlebury.edu`

May 4 Lecture

Textbook Reference:

- *Vector Calculus*, Colley, 4th Edition: §4.3, 4.4

Lagrange Multipliers; Linear Regression

Keywords: the method of Lagrange multipliers, linear regression, polynomial regression

---

Let $f(x,y)$, $g(x,y)$ be differentiable functions. We are interested in the **contstrained optimisation** problem

$$\begin{aligned} \text{maximise/minimise} \quad & f(x,y) \\ \text{subject to} \quad & g(x,y)=0 \end{aligned}$$

Remark: An equation $h(x,y)=c$, c constant, is called a **constraint.** Note that any constraint $h(x,y)=c$ can be rearranged to a constraint of the form $g(x,y)=0$ by letting $g(x,y)=h(x,y)-c$.

In the last lecture we saw that the solutions to this problem - the **constrained extrema** - came in two flavours:

(I) the points (x,y) satisfying $\nabla f(x,y)=0$ and $g(x,y)=0$;

(II) the points (x,y) satisfying $\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=0$, for some nonzero λ, called a **Lagrange multiplier.**

> Note that type (I) points can be considered to be type (II) points for the case $\lambda=0$.

We generalise to the setting of several variables:

Method of Lagrange multipliers: single constraint

Let $f(\underline{x})$, $g(\underline{x})$ be differentiable functions of n variables. If $\underline{x}$ is a solution to the constrained optimisation problem

$$\begin{aligned} \text{maximise/minimise} \quad & f(\underline{x}) \\ \text{subject to} \quad & g(\underline{x})=0 \end{aligned}$$

then there exists some λ such that $(\underline{x},\lambda)$ is a solution to the equation

$$\nabla f(\underline{x})=\lambda\nabla g(\underline{x})$$

Remark: The method of Lagrange multipliers can be extended to the case of multiple constraints $g_1(\underline{x}) = \ldots = g_k(\underline{x}) = 0$. In this case there are two flavours of constrained extrema:

(I) the points $\underline{x}$ satisfying $\nabla f(\underline{x}) = 0$ and $g_1(\underline{x}) = \ldots = g_k(\underline{x}) = 0$;

(II) the points $\underline{x}$ satisfying $\nabla f(\underline{x}) = \sum_{i=1}^{k} \lambda_i \nabla g_i(\underline{x})$ and $g_1(\underline{x}) = \ldots = g_k(\underline{x}) = 0$, for some $\lambda_1, \ldots, \lambda_k$ (not all equal to zero).

The gradient condition states that $\nabla f(\underline{x})$ is orthogonal to the tangent space of the space defined by $g_1 = \ldots = g_k = 0$. For details see p.284 of the textbook.

Example: Model the surface of the Earth by the unit sphere $x^2 + y^2 + z^2 = 1$. A satellite is orbiting the earth at a fixed height - in our model the satellite's orbit is constrained to lie in the sphere $x^2 + y^2 + z^2 = 9$. Assume we are standing at $(1, 0, 0)$ on the surface of the Earth. Let's use Lagrange multipliers to confirm the obvious(?) geometric fact: the satellite is closest to our position when the satellite is at $(3, 0, 0)$.

We model this problem as a constrained optimisation problem:

$$\begin{array}{rl} \text{minimise} & d(x,y,z) = (x-1)^2 + y^2 + z^2 \\ \text{subject to} & g(x,y,z) = x^2 + y^2 + z^2 - 9 = 0 \end{array}$$

Solution:

$$\nabla d = \begin{bmatrix} 2(x-1) & 2y & 2z \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

Solve:

$2(x-1) = 2\lambda x$ ①

$2y = 2\lambda y$ ②

$2z = 2\lambda z$ ③

$x^2+y^2+z^2=9$ ④

$z \neq 0$: ③ $\Rightarrow \lambda = 1$

① $\Rightarrow 2x - 2 = 2x$ ↯

$\Rightarrow \boxed{z = 0}$

$y \neq 0$: ② $\Rightarrow \lambda = 1$

① $\Rightarrow 2x - 2 = 2x$ ↯.

$\Rightarrow \boxed{y = 0}$

④ $\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

Then $(\pm 3, 0, 0)$ are constrained extrema

$d(3,0,0) = 4 < d(-3,0,0)$

$\Rightarrow (3,0,0)$ is minimum.

Applications of Extrema

Linear regression A set S of k points in the plane

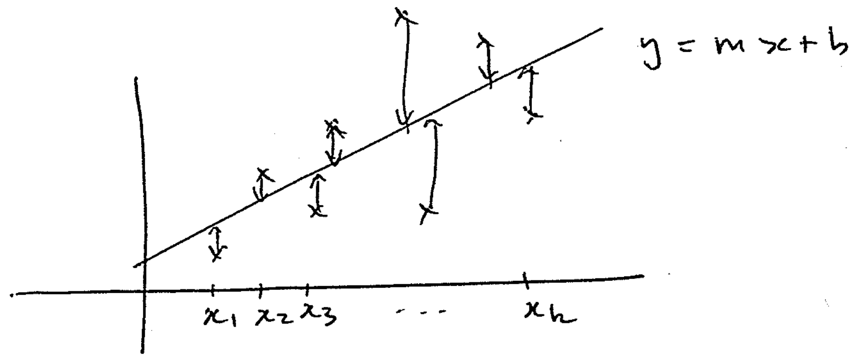
$$S = \{(x_1, y_1), \ldots, (x_k, y_k)\}$$

can be intepreted as a data set relating two quantities x and y. For example, x could represent SAT scores and y could represent college grades.

We want to understand what the general linear correlation is between the quantities x and y i.e. we want to find the **line of best fit** $y = mx + b$. Mathematically, we want to solve the optimisation problem:

$$\text{minimise} \qquad D(m,b) = (y_1 - (mx_1 + b))^2 + \ldots + (y_k - (mx_k + b))^2$$

Diagram:



We need to find the extrema of the function D. We compute ∇D:

$$\frac{\partial D}{\partial m} = \sum_{i=1}^{k} 2\left(y_i - (mx_i + b)\right)\cdot(-x_i) = -2\sum_{i=1}^{k} x_i y_i + 2m\left(\sum x_i^2\right) + 2b\left(\sum x_i\right)$$

$$\frac{\partial D}{\partial b} = \sum 2\left(y_i - (mx_i + b)\right)(-1) = -2\sum y_i + 2m\left(\sum x_i\right) + 2kb$$

Setting both partial derivatives equal to zero gives the equations

$$(\sum x_i^2)m + (\sum x_i)b = \sum x_i y_i$$

$$(\sum x_i)m + kb = \sum y_i$$

This is a system of linear equations in the two variables m and b. We solve to obtain the single solution:

$$m = \frac{k\left(\sum x_i y_i\right) - \left(\sum x_i\right)\left(\sum y_i\right)}{k\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$

$$b = \frac{\left(\sum x_i^2\right)\left(\sum y_i\right) - \left(\sum x_i\right)\left(\sum x_i y_i\right)}{k\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$

This approach can be used to solve more general **polynomial regression**. For example, we could try to determine a **parabola of best fit** $y = ax^2 + bx + c$. Then, we aim to minimise

$$D(a,b,c) = \sum_{i=1}^{k} \left(y_i - (ax_i^2 + bx_i + c)\right)^2$$