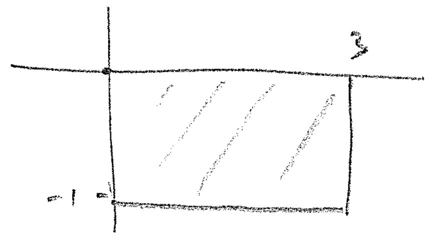
MATH 223 : HW 5/14 SOLUTIONS

1a) $A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

b) $R =$ 

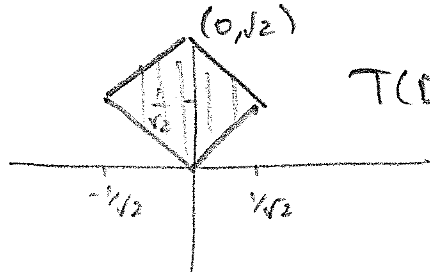
$[3,0] \times [-1,0]$

2). a) Any $\underline{u} = te_1 + se_2 \in D^*$ is transformed to

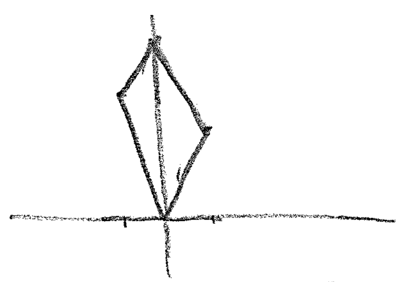
$0 \le t \le 1$
$0 \le s \le 1$

$$T(\underline{u}) = T\left(\begin{bmatrix} t \\ s \end{bmatrix}\right) = \begin{bmatrix} (t-s)/\sqrt{2} \\ (t+s)/\sqrt{2} \end{bmatrix}$$

$$= t\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + s\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$



$T(D^*)$

3) D^* : 

$$T_1(u,v) = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Map $e_1 \longmapsto \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$e_2 \longmapsto \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Then, $T_1(S) = D^*$

Let S be unit square.

Also:

$$x^5(2y-x)e^{(2y-x)^2}$$

$$= (f\circ L)(x,y), \qquad \text{where } f(u,v)=u^5ve^{v^2}$$

$$\Rightarrow \iint_{D'} (f\circ L)\,|\det M|\,dA$$

$$= \iint_{L(D')=D} f\,dA.$$

$$\Rightarrow \int_0^2\int_{x/2}^{x/2+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx$$

$$= \iint_{D'} (f\circ L)\,dA$$

$$= \frac{1}{|\det M|}\cdot\iint_D f\,dA$$

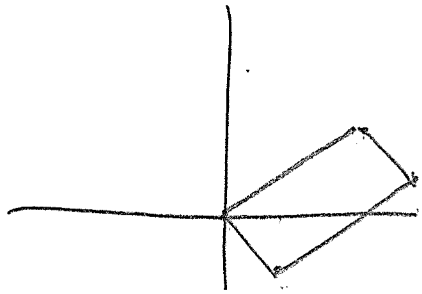
$$= \frac{1}{2}\int_0^2\int_0^2 u^5ve^{v^2}\,du\,dv$$

$$= \frac{1}{2}\int_0^2 \frac{2^6}{6}\cdot ve^{v^2}\,dv = \frac{32}{6}\int_0^2 ve^{v^2}\,dv$$

$$= \frac{32}{6}\left[\frac{1}{2}e^{v^2}\right]_0^2$$

$$= \frac{16}{6}(e^4-1)$$

D :



$$T_2(u,v) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Map $e_1 \longmapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$e_2 \longmapsto \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Then $T_2(S) = D$.

Define $T = T_2 \circ T_1^{-1}$ ie

$$T(\underline{u}) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \underline{u}$$

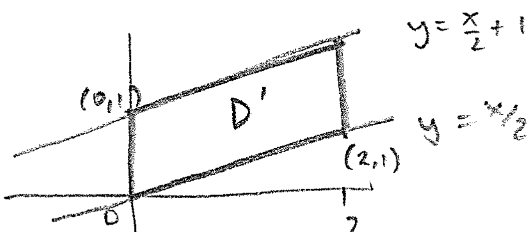
$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \underline{u}$$

$$= \frac{1}{5} \begin{bmatrix} -3 & 4 \\ -7 & 1 \end{bmatrix} \underline{u}.$$

9) Define $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad M$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let D':

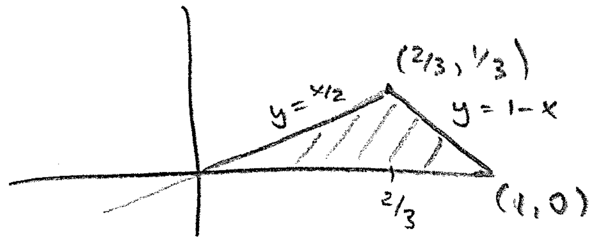


observe: $L(2,1) = (2,0)$ $\Rightarrow$ $L(D') = D$, where

$L(0,1) = (0,2)$ $\qquad D = [2,0] \times [0,2]$

10)

D:



$$\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA$$

Define

$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} u \\ v \end{bmatrix} = \overset{M}{\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

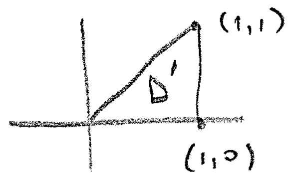
ie $u = x + y$

$v = x - 2y$.

Note $L(2/3, 1/3) = (1, 0)$

$L(1, 0) = (1, 1)$

ie $L(D) =$



Also: $(f \circ L)(x, y) = \sqrt{\frac{x+y}{x-2y}}$

if $f(u, v) = \sqrt{\frac{u}{v}}$.

Hence, by change of coords:

$$\iint_D \sqrt{\frac{x+y}{x-2y}} \cdot |\det M|\, dA$$

$$= \iint_{D'} f\, dA$$

$$\Rightarrow \iint_D \sqrt{\frac{x+y}{x-2y}}\, dA = \frac{1}{|\det M|} \iint_{D'} f\, dA$$

$$= \frac{1}{3} \int_{u=0}^{u=1} \int_{v=0}^{v=u} \sqrt{\frac{u}{v}}\, dv\, du$$

$$= \frac{1}{3} \int_0^1 \sqrt{u} \left[2\sqrt{v} \right]_0^u du$$

$$= \frac{2}{3} \int_0^1 u \, du = \boxed{\frac{1}{3}}$$

12) Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} u \\ v \end{bmatrix} = \overset{M}{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

ie $u = 2x + y$

$v = -x + 2y$

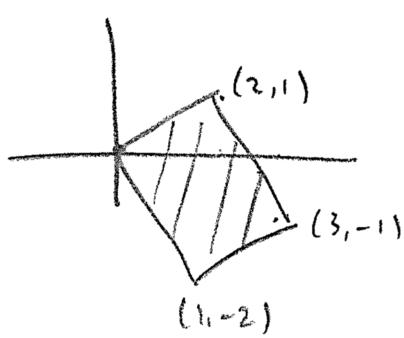
Then,

$$\frac{(2x+y-3)^2}{(2y-x+6)^2} = (f \circ L)(x, y),$$

where $f(u,v) = \frac{(u-3)^2}{(v+6)^2}$

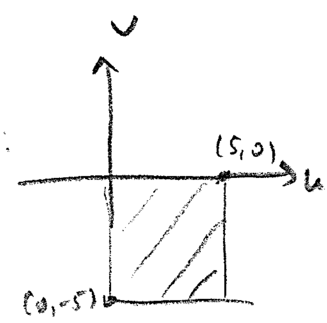
Let

D:



Note: $L(2,1) = (5,0)$ ie $L(D) = D'$:

$L(1,-2) = (0,-5)$



$\Rightarrow$ By change of coordinates:

$$\iint_D \frac{(2x+y-3)^2}{(2y-x+6)^2}\,dx\,dy$$

$$= \frac{1}{|\det M|} \iint_{D'} f\,dA$$

$$= \frac{1}{5} \int_{u=0}^{u=5} \int_{v=-5}^{v=0} \frac{(u-3)^2}{(v+6)^2}\,dv\,du$$

$$= \frac{1}{5} \int_{u=0}^{u=5} \left[-\frac{(u-3)^2}{(v+6)} \right]_{-5}^{0} dv$$

$$= \frac{1}{5} \int_{u=0}^{u=5} \left[(u-3)^2 - \frac{(u-3)^2}{6} \right] du$$

$$= \frac{1}{6} \int_0^5 (u-3)^2\,du$$

$$= \frac{1}{6} \left[\frac{(u-3)^3}{3} \right]_0^5 = \frac{1}{18}\left(2^3 - (-3)^3\right)$$

$$= \boxed{\frac{35}{18}}$$