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March 9 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.1

Functions of several variables

Learning Objectives:
- *Gain familiarity with functions of several variables.*
- *Understand the concept of the graph of a function of several variables.*
- *Gain familiarity with specific examples of surfaces realised as graphs of functions.*
- *Learn the equations defining the quadric surfaces.*

Keywords: *functions of several variables, paraboloid, hyperbolic paraboloid, quadric surfaces*

Functions of several variables

A **function** consists of three pieces of data: a set X (the **domain of** f), a set Y (the **codomain** of f) and a **rule of assignment** that associates to each element $x \in X$ a *unique*, denoted $f(x)$, in the codomain Y. We will use the notation $f : X \to Y$ for a function; when we want to be explicit about the rule, we will write

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x) \end{aligned}$$

The **range of** $f : X \to Y$ is the set of all *outputs* of f i.e.

$$\text{Range}\,(f) = \{y \in Y \mid f(x) = y,\ \text{for some } x \in X\}$$

A function is **surjective (or onto)** if Range $(f) = Y$. A function is **injective (or one-to-one)** if no two distinct elements in X give rise to the same output under f i.e. if $x \neq x'$ are in X then $f(x) \neq f(x')$ in Y.

We are going to study the behaviour of functions $f : X \to Y$ having domain $X \subseteq \mathbb{R}^n$ and codomain $Y \subseteq \mathbb{R}^m$ being subsets of *arbitrary* vector spaces.

Examples:

1. Consider the function
$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (y, 0) \end{aligned}$$
The range of f is the x-axis $\{(c, 0) \mid c \in \mathbb{R}\}$; f is not surjective as it is impossible to find (x, y) so that $f(x, y) = (1, 0)$ i.e. $(1, 0)$ does not lie in the range of f; f is not injective as $f(1, 0) = (0, 0) = f(3, 0)$ i.e. the distinct inputs $(1, 0)$ and $(3, 0)$ have the same output.

2. Consider the function

$$\begin{array}{rcl} f : \mathbb{R}^3 & \rightarrow & \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\} \\ (x, y, z) & \mapsto & \sqrt{x^2 + y^2 + z^2} \end{array}$$

The range of f is set of non-negative real numbers $\{x \in \mathbb{R} \mid x \geq 0\}$; f is surjective because Range (f) equals the codomain; f is not injective because $f(1, 0, 0) = 1 = f(0, 1, 0)$.

Note, for any (x, y, z) on the sphere centred at $(0, 0, 0)$ having radius 1, $f(x, y, z) = 1$.

3. Here is a more interesting example: consider the unit disc D in $\mathbb{R}^2$

$$D = \{(x, y) \mid x^2 + y^2 \leq 0\}$$

D is the set of all points lying inside the unit circle. Suppose that D describes a warm metal plate. Define

$$\begin{array}{rcl} T : \mathbb{R}^3 & \rightarrow & \mathbb{R}_{\geq 0} \\ (x, y, t) & \mapsto & \text{temperature (in Kelvin) at } (x, y) \in D \text{ at time } t \end{array}$$

so that $T(x, y, t)$ is the function defining the temperature of points on the plate at a certain time t. Finding a formula for $T(x, y, t)$ is difficult and involves solving **partial differential equations**. You would be interested in understanding these functions/equations if you are a materials engineer, for example (or you don't want to burn your fingers!).

4. Consider the function

$$\begin{array}{rcl} p : \mathbb{R}^3 & \rightarrow & \mathbb{R} \\ (t, T, w) & \mapsto & \text{relative growth of a plant at time } t\text{, temperature } T\text{, water content } w \end{array}$$

For example, if you over-water the plant (i.e. let w get very large) then you'd expect the plant to exhibit negative relative growth (the plant is dying) - so that $p(t, T, w) < 0$ for w very large.

5. Let $\underline{0} \neq \underline{u} \in \mathbb{R}^3$. Consider the function

$$\begin{array}{rcl} \delta_{\underline{u}} : \mathbb{R}^3 & \rightarrow & \mathbb{R} \\ \underline{x} & \mapsto & \underline{u} \cdot \underline{x} \end{array}$$

The range of $\delta_{\underline{u}}$ is the whole of $\mathbb{R}$: if c is a real number then

$$\delta_{\underline{u}}\left(c\frac{\underline{u}}{|\underline{u}|^2}\right) = \underline{u} \cdot \left(c\frac{\underline{u}}{|\underline{u}|^2}\right) = c\frac{\underline{u} \cdot \underline{u}}{|\underline{u}|^2} = c$$

Hence, $\delta_{\underline{u}}$ is surjective. However, $\delta_{\underline{u}}$ is not injective: let $\underline{x} \neq \underline{0}$ be a vector perpendicular to $\underline{u}$. Then,

$$\delta_{\underline{u}}(\underline{x}) = \underline{u} \cdot \underline{x} = 0 = \underline{u} \cdot (-\underline{x}) = \delta_{\underline{u}}(-\underline{x})$$

6. Define the function

$$\begin{aligned} h :: \mathbb{R}^2 &\to \mathbb{R} \\ P &\mapsto \text{distance between } P \text{ and the line } y + x = 2 \end{aligned}$$

If $P = (x, y)$ then

$$\text{distance between } P \text{ and } y + x = 2 = \frac{\left\| \begin{bmatrix} x \\ y-2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\|}{\sqrt{2}} = \frac{\sqrt{(x+y)^2 - 4(x+y) + 4}}{\sqrt{2}}$$

Hence, we can write

$$h(x, y) = \sqrt{\frac{(x+y)^2 - 4(x+y) + 4}{2}}$$

Graphs of functions

We have already seen a functions of several variables: a *vector field* was a function

$$\underline{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

Given a point $P = \underline{x} \in \mathbb{R}^n$ we associated a vector output

$$\underline{F}(\underline{x}) = \begin{bmatrix} F_1(\underline{x}) \\ \vdots \\ F_n(\underline{x}) \end{bmatrix}$$

where, for each $i = 1, 2, \ldots, n$,

$$\underline{F}_i : \mathbb{R}^n \to \mathbb{R}$$

For example, you considered the vector field

$$\underline{F}(x, y) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

for your homework. Here

$$\underline{F}_1(x, y) = -x., \quad \underline{F}_2(x, y) = y$$

Remark: In general, if we have a function of several variables

$$f : X \subseteq \mathbb{R}^n \to Y \subset \mathbb{R}^m$$

then f assigns to a point $\underline{x} = (x_1, \ldots, x_n) \in X$ the point (or vector)

$$f(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}), \ldots, f_m(\underline{x}))$$

We call the **scalar-valued functions** $f_1, \ldots, f_m : X \to \mathbb{R}$ the **component functions of** f. If the codomain $Y \subseteq \mathbb{R}^m$, $m > 1$, then we say that f is a **vector-valued function**.

We could visually represent a vector field by plotting the output vectors at each point in the domain X. What about for more general functions of several variables? First we focus on scalar-valued functions whose domain X is a subset of $\mathbb{R}^n$.

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The **graph of** f is the subset

$$\{(\underline{x}, x_{n+1}) \in X \times \mathbb{R} \mid x_{n+1} = f(\underline{x})\}$$

For example, consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x, y) \mapsto x^2 + y^2$$

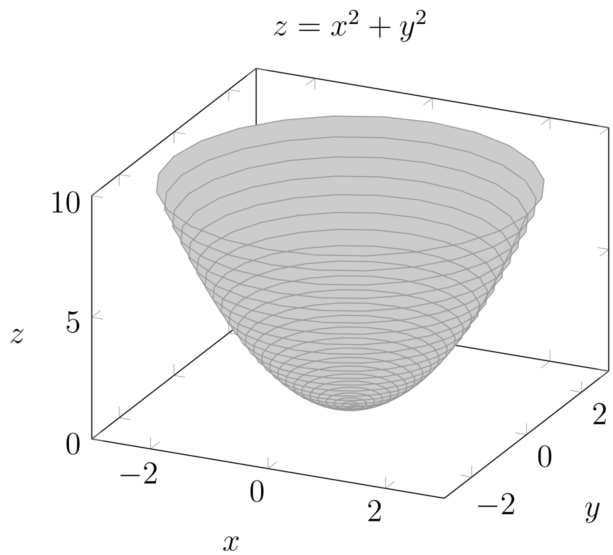
Then, the graph of f is the subset of $\mathbb{R}^3$

$$\{(x, y, x^2 + y^2) \mid x, y, \in \mathbb{R}\}$$

In particular, we see that the graph is the subset of $\mathbb{R}^3$ defined by the equation

$$z = x^2 + y^2$$

We've already identified this surface as a the surface of revolution known as a **paraboloid**.



Next time we will investigate how we can understand the graphs of functions $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$.