Multivariable Calculus
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Contact: `gmelvin@middlebury.edu`

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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §3.1

Kepler's Laws of Planetary Motion (non-examinable)

Today we will make use of the concepts we have seen to derive one of human civilisation's great triumphs: **Kepler's laws of planetary motion**.

Background

Observing the passage of the sun throughout the day, or the passage of the moon and constellations through a clear night sky, it would seem reasonable to make the assumption that the heavenly bodies are revolving around a stationary Earth. This *geocentric* (or *Ptolemaic*[1]) view of the universe was the predominant description of the heavens used by Aristotle through to medieval European and Islamic astronomers. In the 16th Century, Copernicus[2] posited the heretical *heliocentric* view of the universe: the Sun, not the Earth, was the centre of the universe, around which all other heavenly bodies revolved. A common feature of each of these paradigms was the use of *spheres* and *circles* to describe the orbits of the Sun and planets relative to each other.

Figure 1: Tyco Brahe

Figure 2: Johannes Kepler

In the early 17th Century, Johannes Kepler, an assistant of the astronomer Tycho Brahe, analysed Brahe's vast collection of astronomical observations and, in what can be considered one of the first major triumphs of *big data*, conjectured the following:

[1]Claudius Ptolemy, 100-170 CE, a Greco-Roman-Egyptian mathematician and natural philosopher, lived in Alexandria in ancient Egypt in the mid 2nd Century. See `http://en.wikipedia.org/wiki/Ptolemy`

[2]Nicolaus Copernicus, 1473-1543, was a Prussian polymath. Copernicus was wary of publishing his astronomical findings for fear '*he would expose himself on account of the novelty and incomprehensibility of his theses.*' See `http://en.wikipedia.org/wiki/Nicolaus_Copernicus`

Kepler's Laws of Planetary Motion

L1) In the two-body system consisting of the Sun and a planet, the planet's orbit is an **ellipse** and the sun lies at one focus of the ellipse.

L2) During equal intervals of time, a planet sweeps through equal areas with respect to the sun.

L3) If T is the length of time for one planetary orbit, and a is the length of the semimajor axis of this orbit, then $T^2 = Ka^3$, for some constant K.

Remark: it is important to realise that Kepler's determined his laws by *analysing tables of astronomical data* - his Laws were not a consequence of any physical considerations (which is incredible!).

Today we will derive Kepler's First Law as a consequence of Newton's Theory of Gravitation and what we have learned about *parameterised curves* and *spatial geometry*.

Remark: Newton's description of gravity and his '*Laws of Motion*'

Some formulae

Let $\underline{x}(t)$, $\underline{y}(t)$, $\underline{z}(t)$ be differentiable paths in $\mathbb{R}^n$.

(F1) $\frac{d}{dt}\left(\underline{x}(t)\cdot\underline{y}(t)\right) = \underline{x}'(t)\cdot\underline{y}(t) + \underline{x}(t)\cdot\underline{y}'(t)$

(F2) If $|\underline{x}(t)|$ is a constant, for all t, then $\underline{x}(t)\cdot\underline{x}'(t) = 0$

(F3) $\left(\underline{x}\times\underline{y}\right)'(t) = \underline{x}'(t)\times\underline{y}(t) + \underline{x}(t)\times\underline{y}'(t)$

(F4) $(\underline{x}\times\underline{y})\times\underline{z} = (\underline{x}\cdot\underline{z})\underline{y} - (\underline{y}\cdot\underline{z})\underline{x}$

Sketch of proof:

Some physics

Assume the sun is at the origin O in $\mathbb{R}^3$. Let $\underline{r}(t)$ denote the the position at time t of a planet moving under the influence of gravity exerted by the Sun. We write $\underline{v}(t) = \underline{r}'(t)$ for the velocity of $\underline{r}$ and $\underline{a} = \underline{r}''(t)$ for the acceleration of $\underline{r}$.

Goal: use Newton's Laws of Gravity and Motion to determine $\underline{r}(t)$, and therefore describe the motion of the planet.

• **Newton's Law of Gravitation** states that the (attractive) gravitational force $\underline{F}$ experienced by the planet is given by

$$\underline{F} = -GMm\frac{\underline{r}}{|\underline{r}|^3} \tag{1}$$

Here m is the mass of the planet, M is the mass of the Sun, and G is Newton's *gravitational constant* $(= 6.672 \times 10^{-11}\ Nm^2/kg^2)$.

- On the other hand, **Newton's Second Law of Motion** states that

$$\underline{F} = m\underline{a}, \tag{2}$$

Equating (1) and (2) we find

$$m\underline{a} = -GMm\frac{\underline{r}}{|\underline{r}|^3} \quad \Longrightarrow \quad \underline{a} = -GM\frac{\underline{r}}{|\underline{r}|^3} \tag{3}$$

In particular, these physical considerations imply the following

(A) **the acceleration vector $\underline{a}$ is parallel to the position vector $\underline{r}$ of the planet**.

FLEX THOSE MATHEMATICAL MUSCLES!

1. Using $(F3)$ and (A) show that

$$(\underline{r} \times \underline{v})'(t) = \underline{0}$$

   (*Hint: recall that* $\underline{r}' = \underline{v}$)

2. Explain why the path $(\underline{r} \times \underline{v})(t)$ is a *constant path* $\underline{c}$.

3. Deduce that the motion of the planet is planar i.e. the position vector of the planet must lie in a fixed plane containing the Sun.

Kepler's First Law

We will determine a polar equation for $\underline{r}(t)$ and, by changing to Cartesian coordinates, show that this equation is the equation of an ellipse:

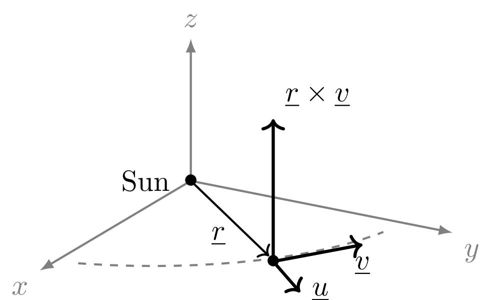
$$\frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$$

By the Exercise above we may assume that the motion of the planet is in a plane, which we may assume is the xy-plane. Hence, we can write $(\underline{r} \times \underline{v})(t) = \underline{c} = c\underline{k}$, where $c \neq 0$ is a constant.

Denote

$$\underline{u}(t) = \frac{\underline{r}(t)}{|\underline{r}(t)|}$$

and write $r(t) = |\underline{r}(t)|$.



We can also compute $\underline{u}$ as follows: since $r(t)\underline{u} = \underline{r}(t)$ we have

$$\underline{v}(t) = \underline{r}'(t) = (r\underline{u})'(t) = r(t)\underline{u}'(t) + r'(t)\underline{u}$$

Hence,

$$\begin{aligned} \underline{c} = \underline{r} \times \underline{v} &= (r(t)\underline{u}(t)) \times [r(t)\underline{u}'(t) + r'(t)\underline{u}(t)] \\ &= r^2(\underline{u} \times \underline{u}') + rr'(\underline{u} \times \underline{u}) \\ &= r^2(\underline{u} \times \underline{u}') \end{aligned} \tag{4}$$

Flex those mathematical muscles

Use (F2) and (F4), and identities (3), (4), together with properties of the cross product, to show that

$$\underline{a} \times \underline{c} = (GM\underline{u})'(t)$$

Recall that G and M are constant.

On the other hand, by (F3), we have

$$(\underline{v} \times \underline{c})'(t) = \underline{a} \times \underline{c}$$

Equating these two expressions for $\underline{a} \times \underline{c}$ gives

$$(GM\underline{u})'(t) = (\underline{v} \times \underline{c})'(t) \qquad \Longrightarrow \qquad \underline{v} \times \underline{c} = GM\underline{u} + \underline{d}$$

where $\underline{d}$ is a constant of integration. Then, since $\underline{v} \times \underline{c}$ and $\underline{u}$ lie in the xy-plane the same is true of $\underline{d}$.

By a coordinate transformation we can assume that $\underline{d} = d\underline{i}$, for some $d \in \mathbb{R}$. Hence,

$$\underline{u} \cdot \underline{d} = d \cos\theta$$

Also,

$$\begin{aligned} c^2 &= |\underline{c}|^2 \\ &= \underline{c} \cdot \underline{c} \\ &= (\underline{r} \times \underline{v}) \cdot \underline{c} \\ &= \underline{r} \cdot (\underline{v} \times \underline{c}) \\ &= (r\underline{u}) \cdot (GM\underline{u} + \underline{d}) \\ &= GMr + rd\cos\theta \end{aligned}$$

Rearranging gives

$$r = \frac{c^2}{GM + d\cos\theta} = \frac{p}{1 + e\cos\theta}$$

where

$$p = \frac{c^2}{GM}, \quad e = \frac{d}{GM}$$

This is the polar equation describing the curve along which the motion of the planet lies.

Flex those mathematical muscles

By writing $r = p - er\cos\theta$, show that this equation is

$$(1 - e^2)x^2 + 2pex + y^2 = p^2$$

in Cartesian coordinates.