Middlebury College

Multivariable Calculus
Spring 2018
Contact: gmelvin@middlebury.edu

March 2 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §3.3

Vector fields and Flow lines

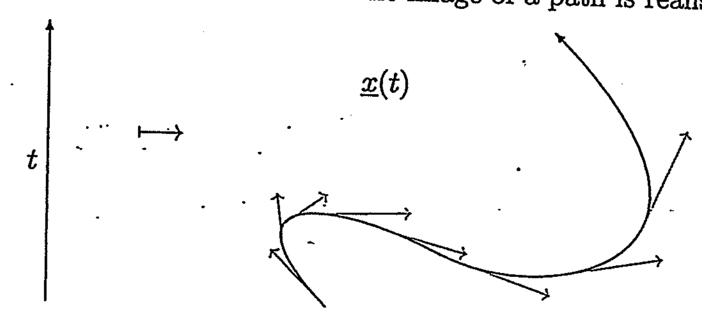
Learning Objectives:
- *Gain familiarity with plotting vector fields.*
- *Understand the concept of a flow line.*
- *Learn how to determine flow lines in simple cases.*

Keywords: *vector fields, flow lines*

In this lecture we consider more general vector-valued functions known as **vector fields**. Vector fields in $\mathbb{R}^2$ or $\mathbb{R}^3$ have natural interpretations in terms of fluid flow, and the mathematical notion of a **flow line** embodies this idea.

Vector fields

In the last lecture we introduced paths as continuous functions $\underline{x} : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an interval of real numbers. The image of a path is realised as a curve:



When $\underline{x}$ is differentiable (so that the component functions of $\underline{x}(t)$ are all differentiable), we can form the velocity vector $\underline{x}'(t_0)$ at $t = t_0$. In this way, we obtain a collection of vectors based at points along the curve described by the path $\underline{x}$.

Keep this example in mind as we go through today's lecture.

A **vector field on** $\mathbb{R}^n$ is a (continuous[1]) function

$$\begin{aligned} \underline{F} : X \subseteq \mathbb{R}^n &\to \mathbb{R}^n \\ (x_1, \dots, x_n) &\mapsto \begin{bmatrix} F_1(x_1, \dots, x_n) \\ \vdots \\ F_n(x_1, \dots, x_n) \end{bmatrix} \end{aligned}$$

which assigns to each *point* in X a *vector* in $\mathbb{R}^n$.

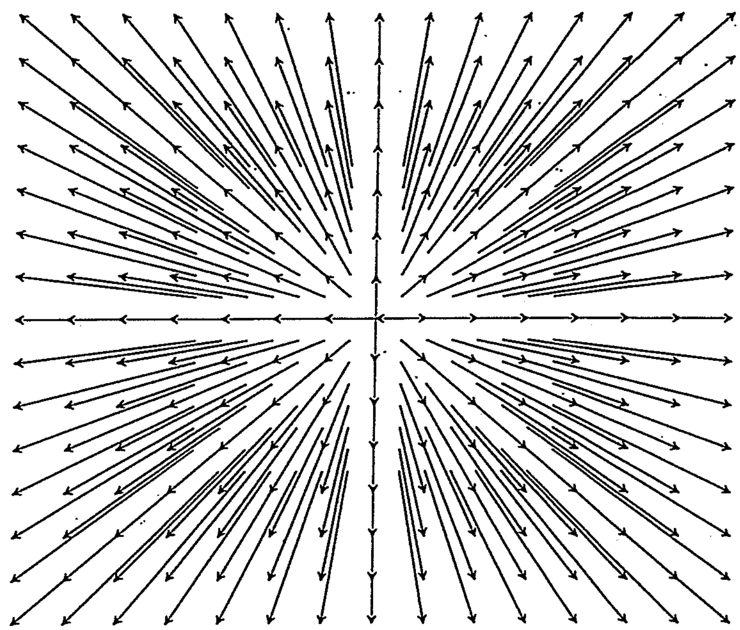
When $n = 2, 3$, we will represent a vector field $\underline{F}$ visually as a collection of vectors based at points in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Example:

---

[1] We will discuss what continuity means for functions of several variables in a couple of weeks.

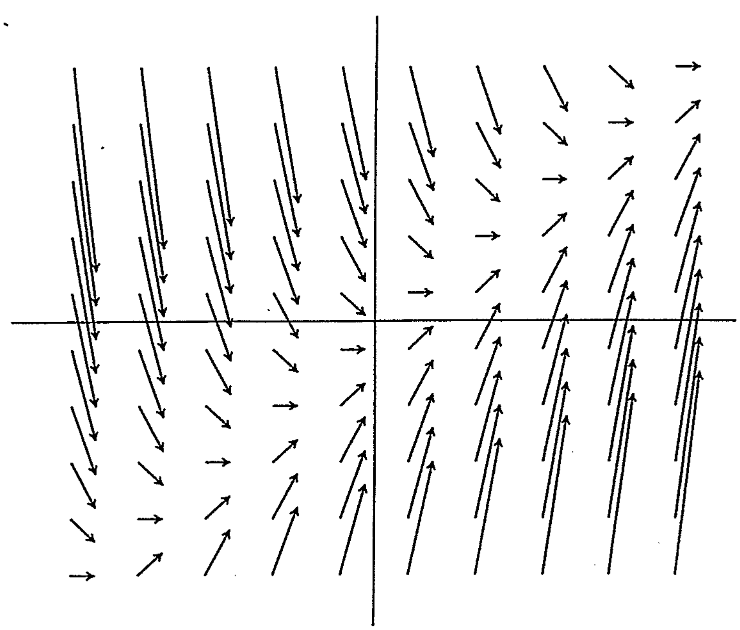
1. Consider the vector field $\underline{F}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}$, defined for every $(x,y) \in \mathbb{R}^2$. We can represent this vector field



General features:

- $|\underline{F}(x,y)|$ grows as $|(x,y)|$ grows
- Vectors in quadrant 1,3 have >0 slope
  " 2,4 have <0 slope

2. Consider the vector field $\underline{F}(x,y) = \begin{bmatrix} 1 \\ (x-y) \end{bmatrix}$, defined for every $(x,y) \in \mathbb{R}^2$. We can represent this vector field



General features:

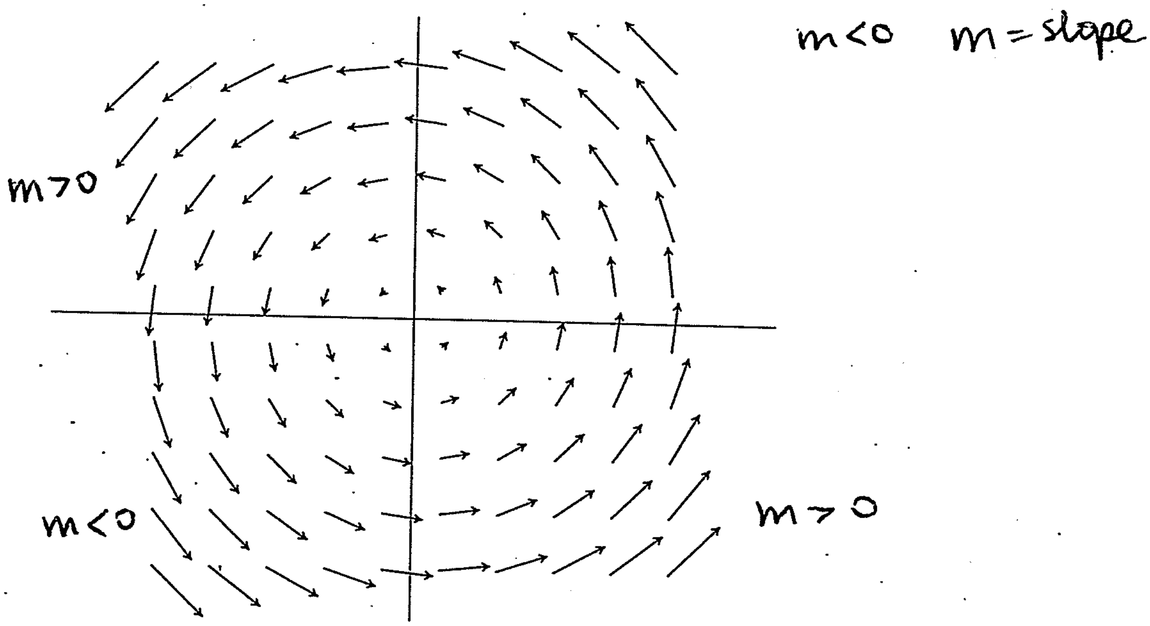
- Constant along lines $y = x + a$
- Vectors always point to the right

Observe that long the line $y = x$ the vector field gives horizontal vectors pointing to the right: when $x = y$, the vector field outputs the vector

$$\underline{F}(x, x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that, for any point (x, x) on the line, we draw the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

3. Consider the vector field $\underline{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$, defined for every $(x, y) \in \mathbb{R}^2$. We can represent this vector field



General features:

- $|F(x,y)|$ grows as $|(x,y)|$ grows.
- $\underline{F}(x,y) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \underline{F}(x,y) \perp$ to $\begin{bmatrix} x \\ y \end{bmatrix}$

Observe that $\underline{F}(x, y) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$, for every (x, y).

Remark: given a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, it can be difficult to represent the vector field visually. However, it's important to think about the general features of the vector field: *Do the vectors point in a general direction in certain regions of the plane/space? Can you identify where the vectors are zero? Can you identify where the vectors have large/small magnitude? Can you identify where the vectors are horizontal/vertical?*

CHECK YOUR UNDERSTANDING

Sketch the given vector fields in $\mathbb{R}^2$.

1. $F(x,y) = \begin{bmatrix} 2 \\ x \end{bmatrix}$

Indpt of y $\Rightarrow$ constant along lines $x = c$

(Examples of)

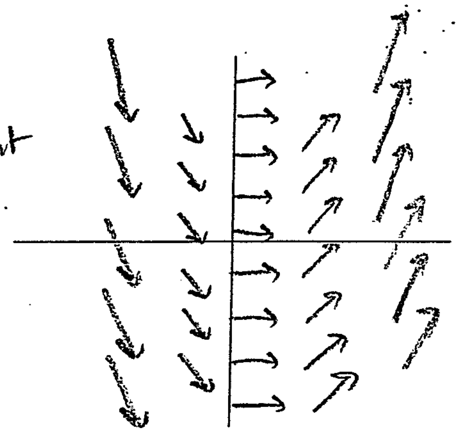
- Flow lines:

$$\underline{x}(t) = \begin{bmatrix} 2t \\ t^2 + a \end{bmatrix}, \quad a \text{ constant}$$

Check:

$$\underline{x}'(t) = \begin{bmatrix} 2 \\ 2t \end{bmatrix}$$

$$\underline{F}(\underline{x}(t)) = \underline{F}\left(\begin{bmatrix} 2t \\ t^2 + a \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2t \end{bmatrix} \checkmark$$



2. $F(x,y) = \begin{bmatrix} x \\ -x \end{bmatrix}$

Indpt of y $\Rightarrow$ constant along lines $x = c$

(Examples of)

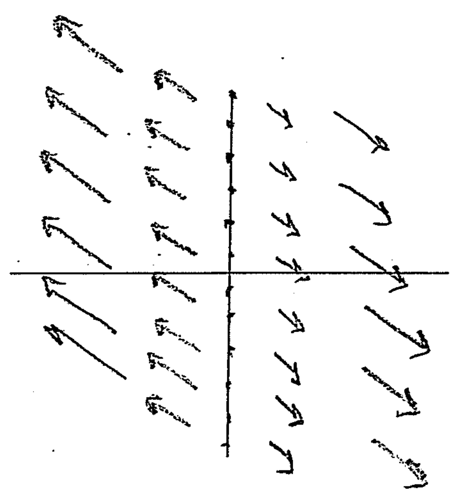
- Flow lines:

$$\underline{x}(t) = \begin{bmatrix} e^t \\ -e^t + a \end{bmatrix}, \quad a \text{ constant}$$

Check:

$$\underline{x}'(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

$$\underline{F}(\underline{x}(t)) = \underline{F}\left(\begin{bmatrix} e^t \\ -e^t + a \end{bmatrix}\right) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$



3. $F(x,y) = \begin{bmatrix} y^2 \\ y \end{bmatrix}$

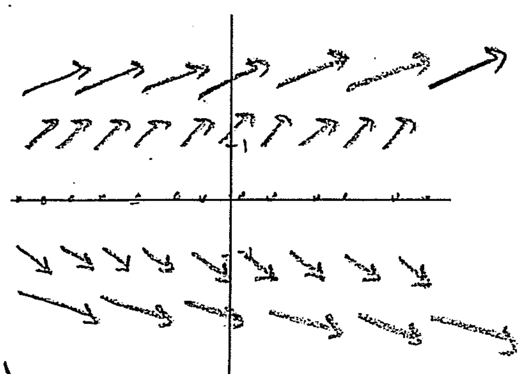
- Indpt of x $\Rightarrow$ constant along lines $y = c$

(Examples of)

Flow lines:

$$\underline{x}(t) = \begin{bmatrix} \frac{1}{2}e^{2t} + a \\ e^t \end{bmatrix}, \quad a \text{ constant}$$

$$\underline{x}'(t) = \begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$$



$$\underline{F}(\underline{x}(t)) = \underline{F}\left(\begin{bmatrix} \frac{1}{2}e^{2t} + a \\ e^t \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$$

Vector fields in nature:

Vector fields arise in lots of places. You've already been looking at vector fields since you were a child:

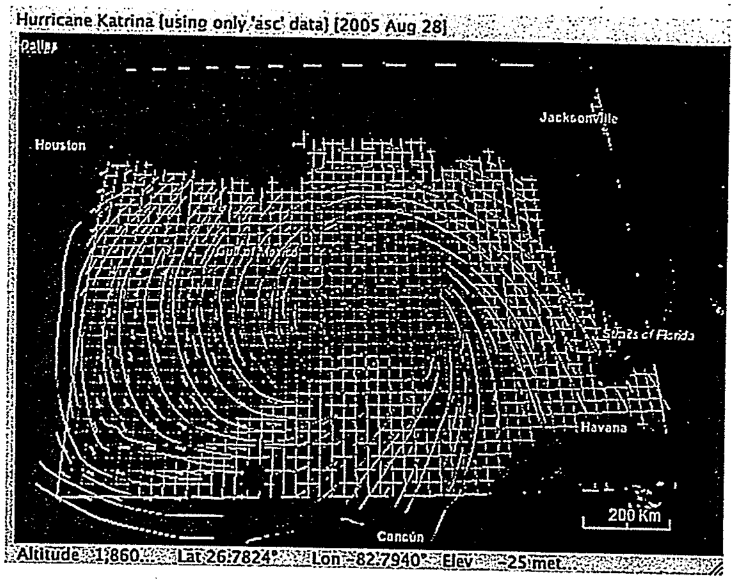


Figure 1: Wind reports are represented as vector fields

Other examples:

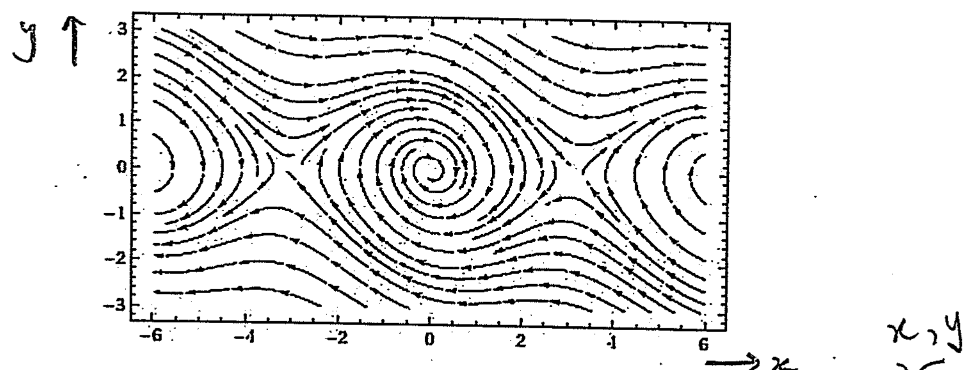


Figure 2: Vector field describing relationship between buy/sell strategy for two stocks in a certain market

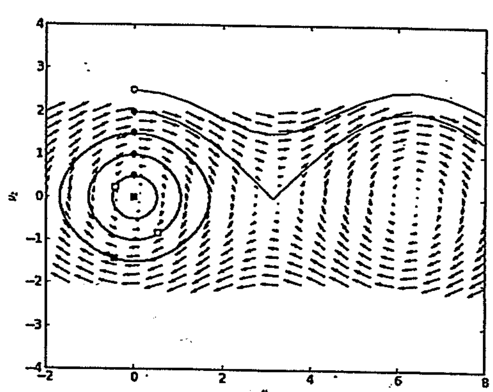


Figure 3: Vector field describing relationship between two stocks with flow lines

Definition: Let $\underline{F}$ be vector field in $\mathbb{R}^n$. A **flow line** of $\underline{F}$ is a differentiable path $\underline{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ satisfying

$$\underline{x}'(t) = \underline{F}(\underline{x}(t)), \quad \text{for every } t \in I.$$

A flow line is a path $\underline{x}$ whose velocity vector at time t is the value of the vector field at the point $\underline{x}(t)$ on the image curve of $\underline{x}$.

Example:

1. Consider the vector field $\underline{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$. The vector field is plotted above.

   The curve $\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ is a flow line for $\underline{F}$: indeed,

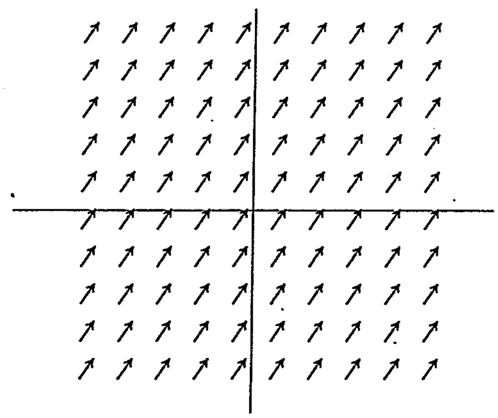
   $$\underline{x}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

   and

   $$\underline{F}(\underline{x}(t)) = \underline{F}\left(\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}\right) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} = \underline{x}'(t), \quad \text{for all } t$$

   Think about the relationship between the path $\underline{x}(t)$ and the general '*shape*' of $\underline{F}$.

2. Consider the vector field $\underline{F}(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The vector field is plotted below.

   A flow line $\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ must satisfy

   $$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \underline{x}'(t) = \underline{F}\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

   That is, we require

   $$x'(t) = 2, \quad y'(t) = 3 \quad \Longrightarrow \quad x(t) = 2t + a, \quad y(t) = 3t + b$$

   Hence, the flow line must take the form

   $$\underline{x}(t) = \begin{bmatrix} 2t + a \\ 3t + b \end{bmatrix}$$

   This is a line in the direction $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ passing through the point (a, b).