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March 21 Lecture

Textbook Reference:

- *Vector Calculus*, Colley, 4th Edition: §2.3, 2.4

Partial Derivatives

Learning Objectives:

- *Understand what it means for a function of several variables to be differentiable.*
- *Learn how to compute the matrix of partial derivatives.*
- *Understand the definition and basic properties of the derivative of a vector-valued function of several variables.*
- *Learn how to compute higher order partial derivatives.*

Keywords: *differentiability, matrix of partial derivatives, the derivative, mixed partial derivatives*

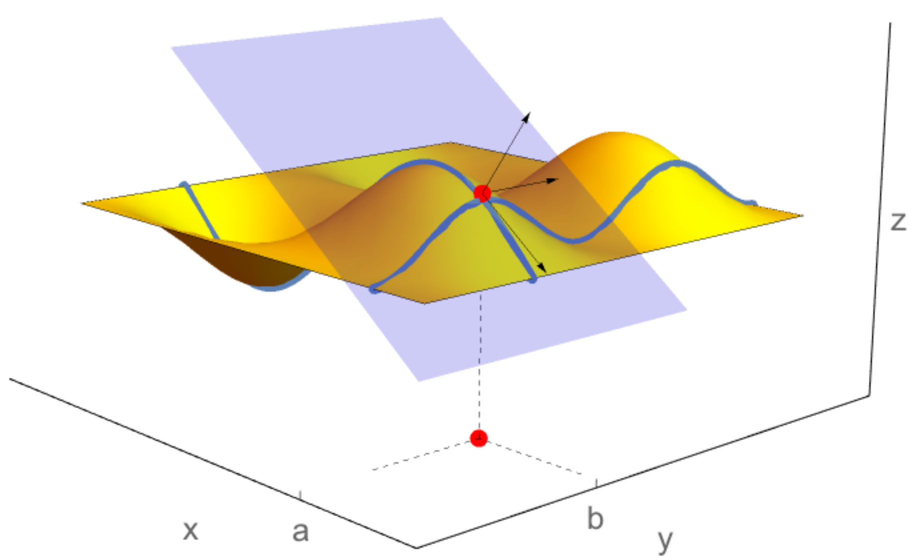
Differentiability

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, $(a, b) \in X$. Suppose that the partial derivatives of f at (a, b) exist. The **linearisation of** f, $L(x, y)$, is the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and the **tangent plane to the graph of** f **at** $(a, b, f(a, b))$ is defined by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \quad (1)$$



Differentiability of $f(x,y)$

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. We say that f **is differentiable at** $\underline{a} = (a,b) \in X$ if the partial derivatives $f_x(a,b)$, $f_y(a,b)$ exist and if

$$\lim_{\underline{x}\to\underline{a}} \frac{f(\underline{x}) - L(\underline{x})}{|\underline{x} - \underline{a}|} = 0$$

If f is differentiable for every $\underline{a} \in X$ then we say that f **is differentiable**.

In words, f **is differentiable at** (a,b) **if** $L(x,y)$ **provides a 'good' approximation of** $f(x,y)$ **near to** (a,b)**.**

Remark:

1. Analytically, '**good**' means that $f(\underline{x}) - L(\underline{x})$ goes to 0 faster than $|\underline{x} - \underline{a}|$.
2. This definition of differentiability extends to scalar-valued functions of n variables $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$.

Example: Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x,y) \mapsto 10 - x^2 - y^2$$

Then, the linearisation of f at $\underline{a} = (a,b)$ is

$$L(x,y) = 10 - a^2 - b^2 - 2a(x-a) - 2b(y-b)$$

We have

$$\begin{aligned} f(x,y) - L(x,y) &= a^2 - x^2 + b^2 - y^2 + 2a(x-a) + 2b(y-b) \\ &= -(x-a)^2 - (y-b)^2 \end{aligned}$$

Then,

$$\frac{f(x,y) - L(x,y)}{|\underline{x} - \underline{a}|} = -\left(\frac{(x-a)^2 + (y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}}\right) = -\sqrt{(x-a)^2 + (y-b)^2}$$

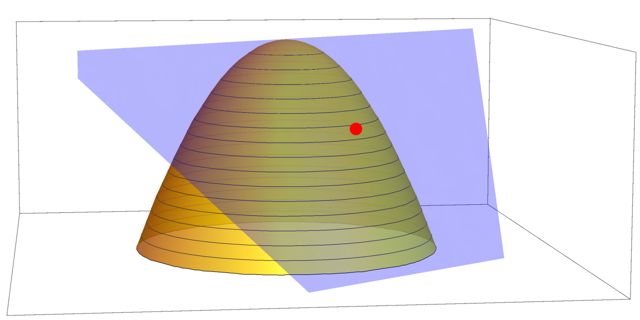
It is now not too difficult to see that

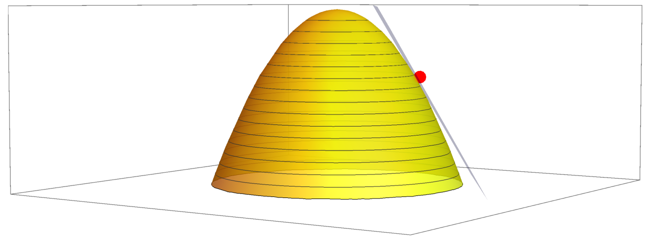
$$\lim_{\underline{x}\to\underline{a}} \frac{f(x,y) - L(x,y)}{|\underline{x} - \underline{a}|} = 0$$

Hence, f is differentiable.
Exercise: show that

$$\lim_{\underline{x}\to\underline{a}} \frac{f(x,y) - L(x,y)}{|\underline{x} - \underline{a}|} = 0$$

using $\epsilon - \delta$ definition.





Remark: Geometrically, f is differentiable if its graph does not have any 'corners'.

Sufficient Condition for differentiability

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, $(a, b) \in X$. If the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous in a sufficiently small disk centred at (a, b) then f is differentiable at (a, b).

Necessary Condition for differentiability

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, $(a, b) \in X$. If f is differentiable at (a, b) then f is continuous at (a, b).

Example:

1. Consider the function $f(x, y) = 2xy + \cos(y^2 + x^2)$. Then,

$$f_x(x, y) = 2y - 2x \sin(y^2 + x^2),$$

$$f_y(x, y) = 2x - 2y \cos(y^2 + x^2).$$

Both the partial derivatives are continuous - use the Algebraic Properties of Continuous Functions (p.111 of Colley). Hence, f is differentiable.

2. Consider the function $f(x, y) = \frac{x^3+5y^4}{1+x^2+y^2}$, defined for all $(x, y) \in \mathbb{R}^2$. Then,

$$f_x(x, y) = \frac{3x^2 + x^4 + 3(xy)^2 - 10xy^4}{(1 + x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{20y^3 + 20x^2y^3 + 10y^5 - 2yx^3}{(1 + x^2 + y^2)^2}$$

Both of those functions are continuous - they are rational functions whose numerator/denominator are polynomial functions are continuous. Hence, $f(x, y)$ is differentiable.

3. Consider the function

$$f(x, y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

The limit does not exist as $(x, y) \to (0, 0)$ (**Exercise!**). Hence, $f(x, y)$ can't be continuous at $(0, 0)$.

However, the partial derivatives

$$\frac{\partial f}{\partial x}(0,0) = \lim h \to 0 \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim h \to 0 \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

do exist. Hence, we see that we require a stronger condition than existence of partial derivatives to ensure differentiability of f at $(0,0)$.

Higher order partial derivatives

It's possible to '**mix and match**' partial derivatives: given a function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ we know how to compute $\frac{\partial f}{\partial x_i}$. We may now compute the partial deriviative of this function with respect to any of the n variables $x_1, \ldots, x_n$.

For example, if $f(x,y,z) = x^2 - 2yz^3 + \frac{3xy^2}{z}$. Then,

$$\frac{\partial f}{\partial x} = 2x + \frac{3y^2}{z}, \quad \frac{\partial f}{\partial y} = -2z^3 + \frac{6xy}{z}, \quad \frac{\partial f}{\partial z} = -6yz^2 - \frac{3xy^2}{z^2}$$

We may now compute the partial derivatives of each of these functions with respect to x, y, z (to obtain a total of nine new functions). We call these functions **second order (or mixed) partial deriviatives of** f:

$$\frac{\partial^2 f}{\partial x^2} \stackrel{def}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} \stackrel{def}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{6y}{z}$$

$$\frac{\partial^2 f}{\partial x \partial z} \stackrel{def}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = -\frac{3y^2}{z^2}, \quad \cdots \text{ etc.}$$

Check your understanding

Compute three of the remaining six second order partial derivatives.

Clairaut's Theorem

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, $(a,b) \in X$. If all first order and second order partial derivatives exist and are continuous then, for any $i, j = 1, \ldots, n$,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

In words, **partial differentiation commutes**.

The Derivative

Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. The **gradient of f at $\underline{a}$** is the (row) vector

$$\nabla f(\underline{a}) = \begin{bmatrix} f_x(\underline{a}) & f_y(\underline{a}) \end{bmatrix}$$

Observation: we can write (1) as

$$z = f(\underline{a}) + \nabla f(\underline{a})(\underline{x} - \underline{a}), \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{1*}$$

The product here is muliplication of the 1×2 matrix $\nabla(f)(\underline{a})$ with the 2×1 matrix $\underline{x} - \underline{a}$.

Remark:

1. Note the analogy with the equation of a tangent line of the graph of a single variable function:
$$y = f(a) + f'(a)(x - a).$$

2. If we consider the change of coordinates
$$\hat{x} = x - a, \quad \hat{y} = y - b, \quad \hat{z} = z - f(a, b)$$
then (1*) becomes
$$\hat{z} = \nabla f(\underline{a})\underline{\hat{x}}, \qquad \underline{\hat{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

3. The above remarks generalise to scalar-valed functions $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, where we define **the gradient of f at $\underline{a}$** to be the $1 \times n$ row vector
$$\nabla f(\underline{a}) = \begin{bmatrix} f_{x_1}(\underline{a}) & f_{x_2}(\underline{a}) & \cdots & f_{x_n}(\underline{a}) \end{bmatrix}$$

Suppose that $\mathbf{f} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is a **vector-valued** function, $\mathbf{f}(\underline{x}) = (f_1(\underline{x}), \ldots, f_m(\underline{x}))$, with each $f_1, \ldots, f_m : X \subseteq \mathbb{R}^n \to \mathbb{R}$ a scalar-valued function.

Define **the matrix of partial derivatives of f at** $\underline{a} \in X$, or **the Jacobian matrix of f at** $\underline{a}$, to be the $m \times n$ matrix $D\mathbf{f}(\underline{a})$ having i^{th} row $\nabla f_i(\underline{a})$:

$$D\mathbf{f}(\underline{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\underline{a}) & \frac{\partial f_1}{\partial x_2}(\underline{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\underline{a}) \\ \frac{\partial f_2}{\partial x_1}(\underline{a}) & \frac{\partial f_2}{\partial x_2}(\underline{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\underline{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\underline{a}) & \frac{\partial f_m}{\partial x_2}(\underline{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\underline{a}) \end{bmatrix}$$

We write $D\mathbf{f}(\underline{x})$, or simple $D\mathbf{f}$, for the $m \times n$ matrix whose ij-entry is $\frac{\partial f_i}{\partial x_j}(\underline{x})$, and call it the **Jacobian of** f.

Define the **linearisation of f at** $\underline{a} \in X$ to be the function

$$\mathbf{L}(\underline{x}) = \mathbf{f}(\underline{a}) + D\mathbf{f}(\underline{a})(\underline{x} - \underline{a}), \quad \underline{x} \in \mathbb{R}^n$$

The product here is multiplication of the $m \times n$ matrix with the $n \times 1$ matrix $\underline{x} - \underline{a}$. In particular, $\mathbf{L}(\underline{x}) \in \mathbb{R}^m$.

Example: Consider the function

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3 \ , \ (x, y) \mapsto (x^2 + y, 2xy, x + y^2)$$

Then,

$$D\mathbf{f}(\underline{x}) = \begin{bmatrix} 2x & 1 \\ 2y & 2x \\ 1 & 2y \end{bmatrix}$$

Differentiability of $\mathbf{f}(\underline{x})$

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. We say that $\mathbf{f}$ **is differentiable at** $\underline{a} \in X$ if all partial derivatives $f_{x_i}(\underline{a})$ exist and if

$$\lim_{\underline{x} \to \underline{a}} \frac{\mathbf{f}(\underline{x}) - \mathbf{L}(\underline{x})}{|\underline{x} - \underline{a}|} = 0$$

If $\mathbf{f}$ is differentiable for every $\underline{a} \in X$ then we say that $\mathbf{f}$ **is differentiable**.

There are analogous results as for the two variable case.

Sufficient Condition for differentiability

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\underline{a} \in X$. If all partial derivatives $f_{x_i}(\underline{x})$ are continuous nearby to $\underline{a}$ then $\mathbf{f}$ is differentiable at $\underline{a}$.

Necessary Condition for differentiability

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\underline{a} \in X$. If $\mathbf{f}$ is differentiable at $\underline{a}$ then $\mathbf{f}$ is continuous at $\underline{a}$.

Moreover, we can reduce differentiability of vector-valued functions to the differentiability of its component functions

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f}(\underline{x}) = (f_1(\underline{x}, \ldots f_m(\underline{x}))$, $\underline{a} \in X$. If $f_1, \ldots, f_m$ are differentiable at $\underline{a}$ then $\mathbf{f}$ is differentiable at $\underline{a}$.

What is the derivative?

Observe the similarity between the linearisation of $\mathbf{f}$ at $\underline{a}$

$$\mathbf{L}(\underline{x}) = \mathbf{f}(\underline{a}) + D\mathbf{f}(\underline{a})(\underline{x} - \underline{a})$$

and function whose graph is the tangent line of a single variable function $f(x)$:

$$L(x) = f(a) + f'(a)(x - a)$$

Define the '*multiplication by* $D\mathbf{f}(\underline{a})$' linear map

$$T_{D\mathbf{f}(\underline{a})} : \mathbb{R}^n \to \mathbb{R}^m \ , \ \underline{x} \mapsto D\mathbf{f}(\underline{a})\underline{x}$$

then the linear map $T_{D\mathbf{f}(\underline{a})}$ plays the role of **the derivative**.

The derivative of a vector-valued function $\mathbf{f}$ of several variables is the linear map defined by the Jacobian of $\mathbf{f}$.