**Contact:** gmelvin@middlebury.edu

# March 19 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.3

## Partial Derivatives

Learning Objectives:
- *Understand the definition of a partial derivative of a function of several variables.*
- *Learn how to compute partial derivatives.*
- *Understand the definition of a linear approximation of a function of two variables.*

Keywords: *partial derivative, tangent plane, linear approximation*

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Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be function of several variables. Define the **partial derivative of $f$ with respect to $x_i$**, denoted $\frac{\partial f}{\partial x_i}$, to be the function of several variables

$$\frac{\partial f}{\partial x_i}(\underline{x}) = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

defined for those $\underline{x} = (x_1, \ldots, x_n) \in X$ such that this limit exists.

*How can we make sense of this definition?*

Consider the function

$$f : \{(x, y) \mid y \neq 0\} \to \mathbb{R}\ ,\ (x, y) \mapsto \frac{3x^2}{y} - xy^2 + 2y$$

Then,

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{3(x+h)^2}{y} - (x+h)y^2 + 2y - \left(\frac{3x^2}{y} - xy^2 + 2y\right)\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{6hx}{y} + \frac{3h^2}{y} - hy^2\right) \\
&= \lim_{h\to 0}\left(\frac{6x}{y} + \frac{3h}{y} - y^2\right) = \frac{6x}{y} - y^2
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y}(x, y) &= \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{3x^2}{y+h} - x(y+h)^2 + 2(y+h) - (\frac{3x^2}{y} - xy^2 + 2y)\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-3x^2h}{y(y+h)} - 2xyh - xh^2 + 2h\right) \\
&= \lim_{h\to 0}\left(\frac{-3x^2}{y(y+h)} - 2xy - xh + 2\right) = \frac{-3x^2}{y^2} - 2xy - x + 2
\end{aligned}$$

**Observation:**

- $\frac{\partial f}{\partial x}(x,y)$ is obtained from $f(x,y)$ by differentiating with respect to $x$ and **treating $y$ as a constant**.
- $\frac{\partial f}{\partial y}(x,y)$ is obtained from $f(x,y)$ by differentiating with respect to $y$ and **treating $x$ as a constant**.

> ## Computation of partial derivatives
>
> Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. To compute $\frac{\partial f}{\partial x_i}$ you should treat all variables $x_j$, $j \neq i$, as constants, and differentiate (as usual) $f$ with respect to $x_i$.

**Example:**

1. Let $f(x,y) = \frac{x}{y}$. Then
$$\frac{\partial f}{\partial x} = \frac{1}{y}, \qquad \frac{\partial f}{\partial y} = \frac{-x}{y^2}$$

2. Let $f(x,y) = \frac{2x^2+y^2}{x^2+y^2}$. Then,
$$\frac{\partial f}{\partial x} = \frac{4x(x^2+y^2) - (2x^2+y^2)2x}{(x^2+y^2)^2} = \frac{2xy^2}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial y} = \frac{2y(x^2+y^2) - (2x^2+y^2)2y}{(x^2+y^2)^2} = \frac{-2yx^2}{(x^2+y^2)^2}$$

3. Let $f(x,y,z) = \sin(x^2z+y) - 2xyz^3 + 5y - 2$. Then,
$$\frac{\partial f}{\partial x} = \cos(x^2z+y)2xz - 2yz^3,$$
$$\frac{\partial f}{\partial y} = \cos(x^2z+y) - 2xz^3 + 5,$$
$$\frac{\partial f}{\partial z} = \cos(x^2z+y)x^2 - 6xyz^2.$$

Check your understanding

Compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, where $f(x,y,z) = \frac{(xyz)^2}{x+y+z}$. *Recall the quotient rule* $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

**Remark:** We will also denote the partial derivative of $f$ with respect to $x_i$, $\frac{\partial f}{\partial x_i}$, by
$$D_{x_i}f(x_1,\dots,x_n) \quad \text{and} \quad f_{x_i}(x_1,\dots,x_n).$$

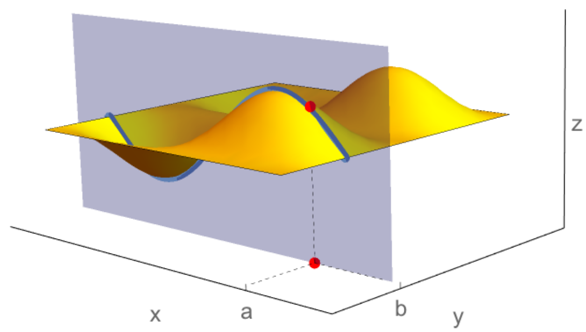
## Geometric interpretation of partial derivatives

Consider a function of two variables $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$, $\underline{a} = (a, b) \in X$.

- Define the single variable function $F_b(x) = f(x, b)$. Then,

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} = \lim_{h\to 0}\frac{F_b(a+h)-F_b(a)}{h} = F_b'(a),$$

  the derivative of $F_b(x)$ at $x = a$, if this limit exists. The graph of the single variable function $F_b(x)$ is the curve obtained by intersecting $\Gamma(f)$, the graph of $f(x, y)$, with the plane $y = b$. Then, $\frac{\partial f}{\partial x}(a, b)$ is the slope of the curve at $(a, b, f(a, b))$.

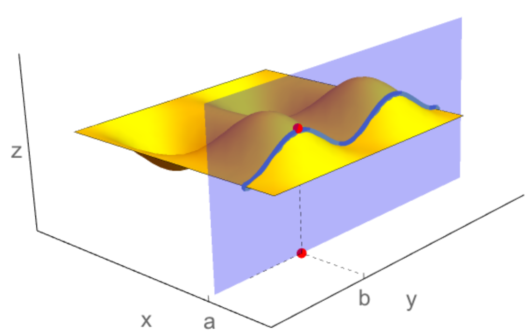



  We may parameterise this curve $\underline{r}_{F_b}(t) = \begin{bmatrix} t \\ b \\ f(t,b) \end{bmatrix}$.

- Define the single variable function $G_a(y) = f(a, y)$. Then,

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h} = \lim_{h\to 0}\frac{G_a(b+h)-G_a(b)}{h} = G_a'(b),$$

  the derivative of $G_a(y)$ at $y = b$, if this limit exists. The graph of the single variable function $G_a(y)$ is the curve obtained by intersecting $\Gamma(f)$, the graph of $f(x, y)$, with the plane $x = a$. Then, $\frac{\partial f}{\partial y}(a, b)$ is the slope of this curve at $(a, b, f(a, b))$.




  We may parameterise this curve $\underline{r}_{G_a}(t) = \begin{bmatrix} a \\ t \\ f(a,t) \end{bmatrix}$.

## Linear Approximations

Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, $\underline{a} = (a, b) \in X$. Suppose that $\frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial f}{\partial y}(a, b)$ are defined (i.e. the limits exist). Define the **linear approximation of $f(x, y)$ at $\underline{a}$** to be the function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The graph $\Gamma(L)$ of $L(x, y)$ is defined by the equation

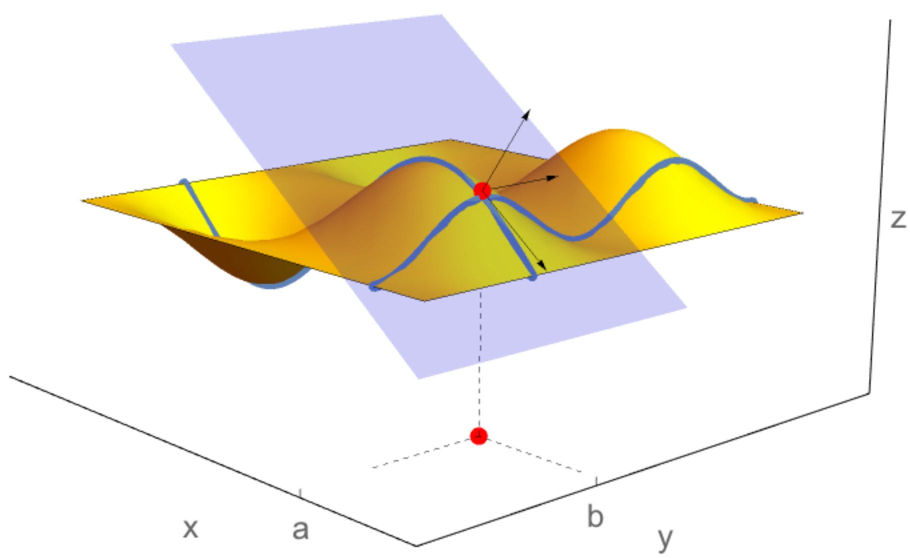
$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

This is the equation of the plane

$$-f_x(a, b)x - f_y(a, b)y + z = f(a, b) - f_x(a, b)a - f_y(a, b)b$$

having normal vector $\underline{n} = \begin{bmatrix} -f_x(a, b) \\ -f_y(a, b) \\ 1 \end{bmatrix}$ and passing through $(a, b, f(a, b))$.

It can be shown that $\underline{n} = \underline{r}'_{F_b}(a) \times \underline{r}'_{G_a}(b)$ (**Exercise!**). In particular, **the graph of the linear approximation $L(x, y)$ is tangent to the graphs of $F_b(x)$ and $G_a(y)$**. We call this plane **the tangent plane to $\Gamma(f)$ at $\underline{a}$** (should it exist).




The following definition will be useful.

**Definition:** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of several variables, $\underline{a} \in X$. Define **the gradient of $f$ at $\underline{a}$** to be the vector (should it exist)

$$\nabla f(\underline{a}) = \begin{bmatrix} f_{x_1}(\underline{a}) \\ \vdots \\ f_{x_n}(\underline{a}) \end{bmatrix}$$

We say '**grad** $f$' for $\nabla f$.

**Example:** Let $f(x, y) = 10 - x^2 - y^2$. Then, the linear approximation of $f(x, y)$ at $(1, 2)$ is

$$L(x, y) = 5 - 2(x - 1) - 4(y - 2)$$

The tangent plane to the graph of $f$ at $(1, 2)$ is plane

$$z = 5 - 2(x - 1) - 4(y - 2) \qquad \Longrightarrow \qquad 2x + 4y + z = 15$$

The tangent plane provides a linear approximation to the graph of $f$ nearby to $(a, b, f(a, b))$. In particular, compute

$$L(0.9, 2.1) = 5 - 2(-0.1) - 4(0.1) = 4.8$$

Compare this with

$$f(0.9, 2.1) = 10 - (0.9)^2 - (2.1)^2 = 4.78$$