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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.2

Limits & Continuity

Learning Objectives:
- *Understand the rigorous definition of a limit of a function of several variables.*
- *Learn the basic properties of limits.*
- *Understand the definition of continuity for a function of several variables.*

Keywords: *limit, continuity*

Rigorous definition of limit

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. We write $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$ if, given any $\epsilon > 0$, you can find $\delta > 0$ such that

$$\text{if } \underline{x} \in X \text{ and } 0 < |\underline{x} - \underline{a}| < \delta \text{ then } |\mathbf{f}(\underline{x}) - \underline{L}| < \epsilon$$

We call $\underline{L}$ the **limit of f as $\underline{x}$ tends to $\underline{a}$**.

Example:

1. We saw last lecture that, if $\mathbf{f}$ is the function

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3\ ,\ (x, y) \mapsto (x, y, 2y)$$

then $\lim_{\underline{x}\to(1,1)} \mathbf{f}(x, y) = (1, 1, 2)$ using the rigorous definition.

2. Define the function

$$p_x : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x, y) \mapsto x$$

If $\underline{a} = (a, b)$ we claim (unsurprisingly) that $\lim_{\underline{x}\to\underline{a}} p_x(x, y) = a$: indeed, suppose we are given $\epsilon > 0$. Take $\delta = \epsilon$. Then, for any $\underline{x} = (x, y)$ satisfying

$$0 < |\underline{x} - \underline{a}| = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

we find

$$|p_x(x, y) - a| = |x - a| < \delta = \epsilon.$$

Here we use that, when $x \neq a$,

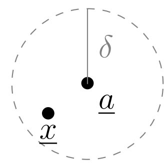
$$\sqrt{(x-a)^2 + (y-b)^2} = |x-a|\sqrt{1 + \frac{(y-b)^2}{(x-a)^2}} \geq |x-a|$$

Similarly, we could define p_y and p_z (or, more generally p_{x_i}) and obtain analogous results. (**Exercise:** formulate and prove these results).

Remark: You will have seen a similar definition for the limit of a single variable function in Calculus I.

Determining limits of several variable functions

In general, checking the $\epsilon - \delta$ definition gets very messy very quickly.
Observation: Let $\underline{a} \in \mathbb{R}^n$, where $n = 2, 3$. If $\underline{x} \in \mathbb{R}^n$ satisfies $|\underline{x} - \underline{a}| < \delta$ then $\underline{x}$ lies inside the disc/sphere of radius δ, centred at $\underline{a}$.



Therefore, the statement $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$ means that, as $\underline{x}$ moves towards $\underline{a}$, $\mathbf{f}(\underline{x})$ moves towards $\underline{L}$, *irrespective of the path $\underline{x}$ takes to get close to $\underline{a}$.*

Example:

1. Consider the function

$$f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R} \ , \ (x,y) \mapsto \frac{2x^2 + y^2}{x^2 + y^2}$$

This function is not defined at $(0,0)$ - we can't evaluate the quantity $\frac{0}{0}$. However, we could still ask whether $\lim_{\underline{x}\to(0,0)} \mathbf{f}(\underline{x})$ exists.

If this limit did exist then it will be the same no matter how we approach $(0,0)$. For example, if we approach $(0,0)$ along the x-axis, where $y = 0$, then

$$\mathbf{f}(x, 0) = \frac{2x^2 + 0}{x^2 + 0} = 2, \quad x \neq 0$$

This means that the function $\mathbf{f}$ is constant along the x-axis. Similarly, if we approach 0 along the y-axis, where $x = 0$, then

$$\mathbf{f}(0, y) = \frac{0 + y^2}{0 + y^2} = 1 \quad y \neq 0$$

We see that approaching $(0,0)$ from two different directions gives rise to two distinct values for the limit. Therefore, $\lim_{\underline{x}\to(0,0)} \mathbf{f}(\underline{x})$ does not exist.

2. We can also use different coordinate systems: consider the function

$$g : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R} \ , \ (x,y) \mapsto \frac{x^2 - y^2}{x^2 + y^2}$$

In polar coordinates we have

$$g(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

Evaluating $\lim_{(x,y)\to(0,0)} g(x,y)$ is the same as understanding the behaviour of $g(r\cos\theta, r\sin\theta)$ as $r \to 0$: in particular, there is no restriction on θ. Moreover, if this limit exists then it should be *independent of* θ. However, we see that

$$\lim_{r\to 0} g(r\cos\theta, r\sin\theta) = \lim_{r\to 0}\cos 2\theta = \cos 2\theta$$

so that the limit can't possibly exist.

Exercise: how does this approach using polar coordinates relate to determining the limit by approaching $(0,0)$ along the x and y axes?

3. We can also use different coordiante systems to **determine limits**: consider the function

$$h : \mathbb{R}^2 - \{(0,0\} \to \mathbb{R}\ ,\ (x,y) \mapsto \frac{x^4 + 3y^5}{x^2 + y^2}$$

Then, in polar coordinates

$$h(r\cos\theta, r\sin\theta) = r^2(\cos^4\theta + 3r\sin^5\theta)$$

Then, we certainly have

$$-3r \leq \cos^4\theta + 3r\sin^5\theta \leq 1 + 3r$$

$$\implies \quad -3r^3 \leq h(r\cos\theta, r\sin\theta) \leq r^2 + 3r^3$$

Thus, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{r\to 0} h(r\cos\theta, r\sin\theta) = 0$$

Properties of Limits

Determining the limit of a function of several variables using the rigorous definition is difficult. This is why Theorems are helpful: they are tools that make life easier for us. An intuitively obvious, but very important, result is the following:

Theorem: (Uniqueness of Limits) If a limit exists then it is unique.

We also have the following:

Algebraic Properties of Limits

1. If $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(x) = \underline{L}$ and $\lim_{\underline{x}\to\underline{a}} \mathbf{g}(x) = \underline{M}$ then $\lim_{\underline{x}\to\underline{a}} (\mathbf{f}+\mathbf{g})(x) = \underline{L} + \underline{M}$.

2. If $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(x) = \underline{L}$ then $\lim_{\underline{x}\to\underline{a}} (k\mathbf{f})(x) = k\underline{L}$, for any scalar k.

3. If f, g are scalar-valued functions, $\lim_{\underline{x}\to\underline{a}} f(x) = L$ and $\lim_{\underline{x}\to\underline{a}} g(x) = M$, with $M \neq 0$, then $\lim_{\underline{x}\to\underline{a}} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$.

Example: A **polynomial function in** x**,** y is a function of the form (e.g.)

$$f(x,y) = x^2y^3 - 10y^3 + x^3y^2 + x + 2$$

If $\underline{a} = (a,b)$ then for a polynomial function $f(x,y)$

$$\lim_{\underline{x}\to\underline{a}} f(x,y) = f(a,b)$$

To see this for the example above, note the equality of functions (where p_x, p_y are defined above)

$$f(x,y) = ((p_x)^2(p_y)^3 - 10(p_y)^3 + (p_x)^3(p_y)^2 + p_x + 2)(x,y) \qquad (*)$$

Since we've shown that $\lim_{\underline{x}\to\underline{a}} p_x(x,y) = a$, and it's not too hard to show that $\lim_{\underline{x}\to\underline{a}} p_y(x,y) = b$, we can use the Algebraic Properties and $(*)$ to obtain

$$\lim_{\underline{x}\to\underline{a}} f(x,y) = a^2b^3 - 10b^3 + a^3b^2 + a + 2 = f(a,b)$$

There's an analogous notion of a **polynomial function** $f(\underline{x})$ **in** n **variables** $\underline{x} = (x_1, \ldots x_n)$.

Continuity

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function of several variables, $\underline{a} \in X$. We say that **f is continuous at** $\underline{a}$ if

$$\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \mathbf{f}(\underline{a})$$

We say that **f is continuous** if it is continuous at $\underline{a}$, for every $\underline{a} \in X$.

Remark: 1. A scalar-valued function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is continuous if its graph $\Gamma(f)$ admits no jumps/breaks/gaps.

2. The Algebraic Properties of Limits translate into Algebraic Properties of Continuous Functions (see p.111 of Colley).

Example:

1. A polynomial function $f(\underline{x})$ in n variables is continuous.

2. The function
$$f : \mathbb{R}^2 \to \mathbb{R}\ , \begin{cases} \frac{2x^2+y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$
is not continuous at $(0,0)$: $\lim_{\underline{x}\to(0,0)} f(x,y)$ does not even exist.

3. The function
$$g : \mathbb{R}^2 \to \mathbb{R}\ , \begin{cases} \frac{x^4+3y^5}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$
is continuous at $(x,y) = (0,0)$: we've shown above that $\lim_{\underline{x}\to(0,0)} g(x,y) = 0 = g(0,0)$. In fact, this function is continuous (i.e. continuous at all $\underline{a} \in \mathbb{R}^2$).

The following result reduces the analysis of vector-valued continuous functions to that of scalar-valued continuous functions.

Theorem: Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a functions of several variables, $\mathbf{f}(\underline{x}) = (f_1(\underline{x}), \ldots, f_m(\underline{x}))$. Then, $\mathbf{f}$ is continuous at $\underline{a} \in X$ if and only if the component functions $f_1(\underline{x}), \ldots f_m(\underline{x})$ are continuous at $\underline{a} \in X$.