# March 14 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.2

## Limits & Continuity

Learning Objectives:
- *Understand the concept of limit for a function of several variables.*
- *Learn how to determine limits for simple functions.*

Keywords: *limit*

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Today we introduce the notion of a limit for a function of several variables. We will introduce the intuitive notion of a limit and see how to determine the limit of some rational functions. We will define what it means for a function of several variables to be continuous.

## Limits of functions of several variables

Let

$$\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m \ , \ \underline{x} \mapsto \mathbf{f}(\underline{x}) = (f_1(\underline{x}), \dots, f_m(\underline{x}))$$

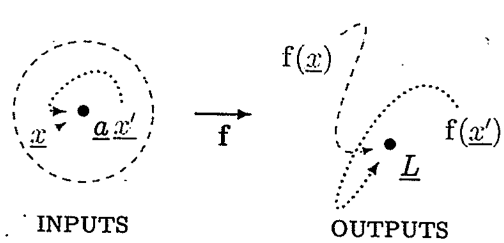
be a function of several variables.

### Intuitive notion of limit I

Intuitively, the **limit of f as $\underline{x}$ tends to $\underline{a}$** is the vector $\underline{L} \in \mathbb{R}^m$ that $\mathbf{f}(\underline{x})$ *approaches* whenever $\underline{x}$ *is near to* $\underline{a}$ (but not equal to $\underline{a}$), should such a vector $\underline{L}$ exist.

In the case that $\underline{L}$ exists, we write

$$\lim_{\underline{x} \to \underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$$




We need to be more precise with what we mean by *approaches* and *near to*.

## Intuitive notion of limit II

Intuitively,

$$\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$$

means that we can make $|\mathbf{f}(\underline{x}) - \underline{L}|$ arbitrarily small (i.e. as close to 0 as we please) by keeping $|\underline{x} - \underline{a}|$ sufficiently small (but nonzero).

**Example:** Consider the function

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3 \ , \ (x, y) \mapsto (x, y, 2y)$$

Intuitively, as $\underline{x} = (x, y)$ gets close, but not equal, to $\underline{a} = (1, 1)$ we expect that $\mathbf{f}(\underline{x})$ gets close to $\underline{L} = (1, 1, 2)$: for $\underline{x}$ such that

$$\left| \underline{x} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right| = \sqrt{(x-1)^2 + (y-1)^2}$$

is sufficiently small (i.e. $\underline{x}$ is sufficiently close to $(1, 1)$), we find that we can make

$$\left| \mathbf{f}(\underline{x}) - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right| = \sqrt{(x-1)^2 + (y-1)^2 + (2y-2)^2} = \sqrt{(x-1)^2 + 5(y-1)^2}$$

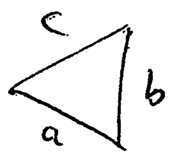
arbitrarily small.

For example, to make $|\mathbf{f}(\underline{x}) - \underline{L}| < 0.01$ we can take those $\underline{x} \in \mathbb{R}^2$ such that $|\underline{x} - \underline{a}| < \frac{1}{1000} = 0.001$: indeed, in this case

$$\begin{aligned} |\mathbf{f}(\underline{x}) - \underline{L}| &= \sqrt{(x-1)^2 + 5(y-1)^2} \\ &\leq |x-1| + \sqrt{5}|y-1| \quad \leftarrow \text{Triangle inequality} \\ &\leq 5|x-1| + 5|y-1| \\ &< \frac{5}{1000} + \frac{5}{1000} \quad \leftarrow \text{by } * \\ &= \frac{1}{100} = 0.01 \end{aligned}$$

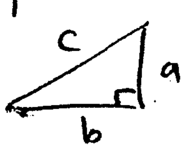
**Notes:**

- Triangle equality:

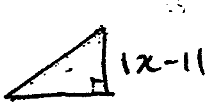



$c \leq a + b$

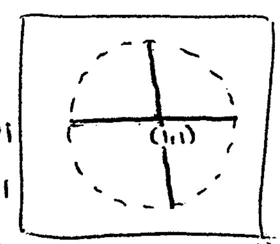
In particular,




$c = \sqrt{a^2 + b^2} \leq a + b$




- (*) If $|\underline{x} - \underline{a}| < 0.001$ then $\underline{x}$ lives in disc of radius 0.001 having ~~radius~~ centre $\underline{a}$




$\Rightarrow$

$1 - 0.001 < x < 1 + 0.001$

$1 - 0.001 < y < 1 + 0.001$

$\rightarrow |x - 1| < 0.001 = \frac{1}{1000}$

$\rightarrow |y - 1| < 0.001 = \frac{1}{1000}$

CHECK YOUR UNDERSTANDING

1. Determine $\delta > 0$ such that $|\mathbf{f}(\underline{x}) - \underline{L}| < \frac{1}{500}$ whenever $|\underline{x} - \underline{a}| < \delta$.

Take $\delta = 0.002 = \frac{1}{5000}$. Then,

$$|\underline{f}(\underline{x}) - \underline{L}| \leq 5|x-1| + 5|y-1|$$
$$< \frac{5}{5000} + \frac{5}{5000} = \frac{10}{5000} = \frac{1}{500}$$

2. Let $\epsilon > 0$. Determine $\delta > 0$ such that $|\mathbf{f}(\underline{x}) - \underline{L}| < \epsilon$ whenever $|\underline{x} - \underline{a}| < \delta$.

Take $\delta = \frac{\varepsilon}{10}$. Then

$$|\underline{f}(\underline{x}) - \underline{L}| \leq 5|x-1| + 5|y-1|$$
$$< 5\delta + 5\delta = 10\delta = \varepsilon.$$

**Rigorous definition of limit**

Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function. We write $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$ if, given any $\epsilon > 0$, you can find $\delta > 0$ such that

$$\text{if } \underline{x} \in X \text{ and } 0 < |\underline{x} - \underline{a}| < \delta \text{ then } |\mathbf{f}(\underline{x}) - \underline{L}| < \epsilon$$

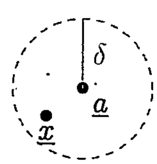
We call $\underline{L}$ the **limit of f as $\underline{x}$ tends to $\underline{a}$**.

**Remark:** You should have seen a similar definition for the limit of a single variable function in Calculus I.

## Determining limits of several variable functions

In general, verifying the $\epsilon - \delta$ condition above gets very messy very quickly. One important observation is the following:

**Observation:** Let $\underline{a} \in \mathbb{R}^n$, where $n = 2, 3$. If $\underline{x} \in \mathbb{R}^n$ satisfies $|\underline{x} - \underline{a}| < \delta$ then $\underline{x}$ lies inside the disc/sphere of radius $\delta$, centred at $\underline{a}$.




Therefore, the statement $\lim_{\underline{x}\to\underline{a}} \mathbf{f}(\underline{x}) = \underline{L}$ means that, as $\underline{x}$ moves towards $\underline{a}$, $\mathbf{f}(\underline{x})$ moves towards $\underline{L}$, irrespective of the path $\underline{x}$ takes to get close to $\underline{a}$.

**Example:** Consider the function

$$f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}\ ,\ (x,y) \mapsto \frac{2x^2+y^2}{x^2+y^2}$$

This function is not defined at $(0,0)$ - we can't evaluate the quantity $\frac{0}{0}$. However, we could still ask whether $\lim_{\underline{x}\to(0,0)} \mathtt{f}(\underline{x})$ exists.

If this limit did exist then it will be the same no matter how we approach $(0,0)$. For example, if we approach $(0,0)$ along the $x$-axis, where $y=0$, then

$$\mathtt{f}(x,0) = \frac{2x^2+0}{x^2+0} = 2, \quad x \neq 0$$

This means that the function $\mathbf{f}$ is constant along the $x$-axis. Similarly, if we approach 0 along the $y$-axis, where $x=0$, then

$$\mathtt{f}(0,y) = \frac{0+y^2}{0+y^2} = 1 \quad y \neq 0$$

We see that approaching $(0,0)$ from two different directions gives rise to two distinct values for the limit. Therefore, $\lim_{\underline{x}\to(0,0)} \mathtt{f}(\underline{x})$ does not exist.