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March 12 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.1

Functions of several variables

Learning Objectives:
- *Understand the concept of the graph of a function of several variables.*
- *Gain familiarity with specific examples of surfaces realised as graphs of functions.*
- *Gain familiarity with other surfaces in $\mathbb{R}^3$, the quadric surfaces.*

Keywords: *graphs, paraboloid, hyperbolic paraboloid, quadric surfaces*

Graphs of functions

We have already seen a functions of several variables: a *vector field* was a function

$$\underline{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

Given a point $P = \underline{x} \in \mathbb{R}^n$ we associated a vector output

$$\underline{F}(\underline{x}) = \begin{bmatrix} F_1(\underline{x}) \\ \vdots \\ F_n(\underline{x}) \end{bmatrix}$$

where, for each $i = 1, 2, \ldots, n$,

$$\underline{F}_i : \mathbb{R}^n \to \mathbb{R}$$

For example, you considered the vector field

$$\underline{F}(x, y) = \begin{bmatrix} -x \\ y \end{bmatrix}$$

for your homework. Here

$$\underline{F}_1(x, y) = -x., \quad \underline{F}_2(x, y) = y$$

Remark: In general, if we have a function of several variables

$$f : X \subseteq \mathbb{R}^n \to Y \subset \mathbb{R}^m$$

then f assigns to a point $\underline{x} = (x_1, \ldots, x_n) \in X$ the point (or vector)

$$f(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}), \ldots, f_m(\underline{x}))$$

We call the **scalar-valued functions** $f_1, \ldots, f_m : X \to \mathbb{R}$ the **component functions of** f. If the codomain $Y \subseteq \mathbb{R}^m$, $m > 1$, then we say that f is a **vector-valued function**.

We could visually represent a vector field by plotting the output vectors at each point in the domain X. What about for more general functions of several variables? First we focus on scalar-valued functions whose domain X is a subset of $\mathbb{R}^n$.

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. The **graph of** f is the subset

$$\Gamma(f) = \{(\underline{x}, x_{n+1}) \in X \times \mathbb{R} \mid x_{n+1} = f(\underline{x})\}$$

For example, consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x, y) \mapsto x^2 + y^2$$

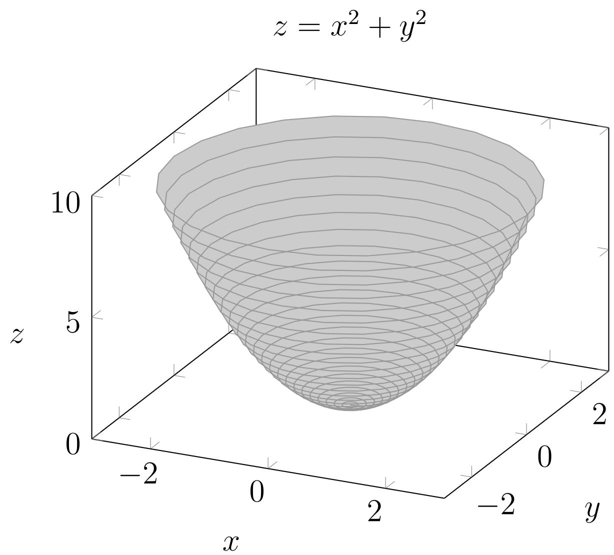
Then, the graph of f is the subset of $\mathbb{R}^3$

$$\{(x, y, x^2 + y^2) \mid x, y, \in \mathbb{R}\}$$

In particular, we see that the graph is the subset of $\mathbb{R}^3$ defined by the equation

$$z = x^2 + y^2$$

We've already identified this surface as a the surface of revolution known as a **paraboloid**.



Remark: A surface S in $\mathbb{R}^3$ is the graph of some function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ if S can be described by an equation of the form $z = f(x, y)$, $(x, y) \in X$.

In your first course in calculus, there was a tight interaction between the behaviour of a function (e.g. increasing? decreasing?), its graph, and the derivative. We are going to generalise this interaction to functions of several variables.

First, we need to understand how to determine the graph of a function.

Basic Question:

How can we 'draw' the graph of a function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$?

Let's look at an example. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R} \; ; \; (x, y) \mapsto y + 2x^2$$

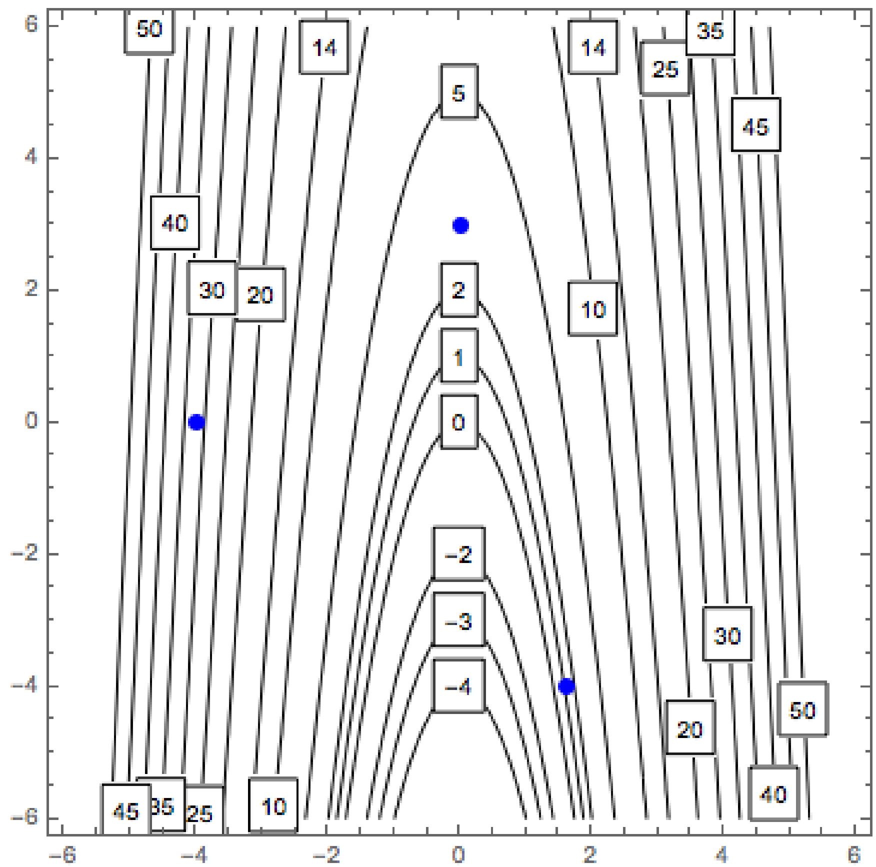
We are going to describe an approach to better understand the graph of f

$$\Gamma(f) = \{(x, y, z) \mid z = f(x, y)\} = \{(x, y, z) \mid z = y + 2x^2\}$$

Observation: since $\Gamma(f)$ is the graph of a function it must pass the **vertical line test** - for any point (x, y) in the domain of f (in this, the domain is the whole of $\mathbb{R}^2$), the vertical line passing through (x, y) intersects $\Gamma(f)$ in a single point lying above/below (x, y) at height $z = f(x, y)$.

Let's consider all those points (x, y) for which $f(x, y) = 0$ - this will correspond to that part of $\Gamma(f)$ at height $z = 0$ i.e. the intersection of $\Gamma(f)$ with the xy-plane. Of course, this is the collection of points (x, y) such that $y = -2x^2$.

We plot this collection of points in $\mathbb{R}^2$ and call this curve a **level curve of** f **at height** $z = 0$ (or **level set of** f **at height** $z = 0$). We plot below further level curves of f at heights $z = -4, -3, -2, 1, 2, 5, 10, 14, 20, 25, 30, 35, 40, 45, 50$



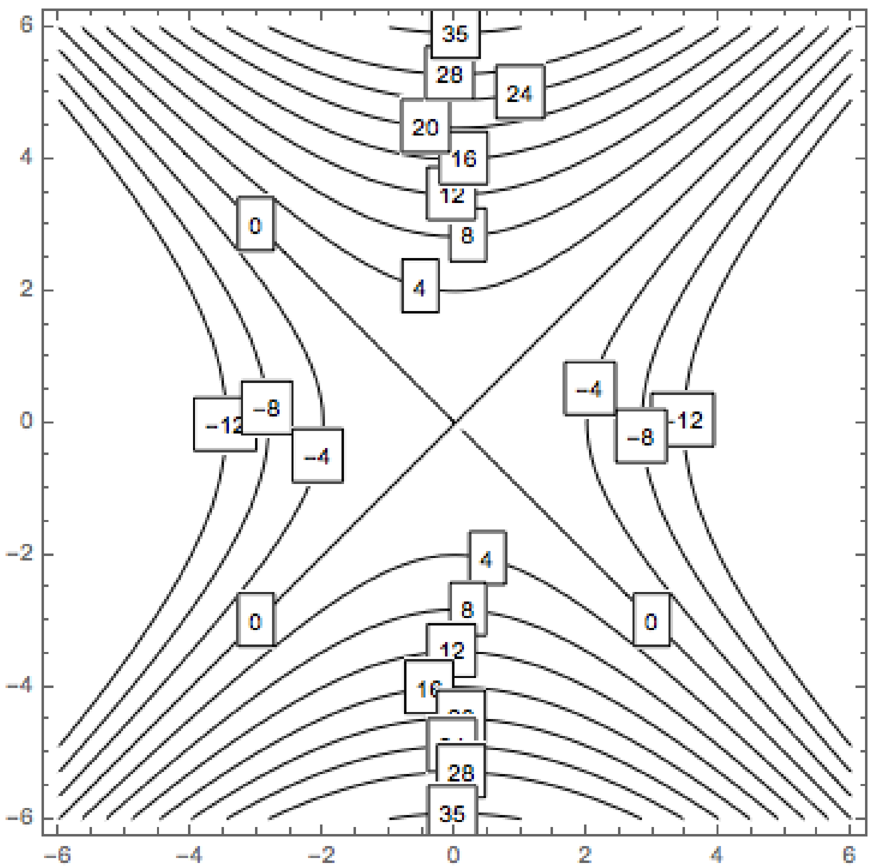
Observations:

Remark: The level curve of f at height $z = c$ is obtained by projecting the intersection of $\Gamma(f)$ with the plane $z = c$ onto the xy-plane.

We plot the level curves of the function

$$h : \mathbb{R}^2 \to \mathbb{R} \ , \ (x, y) \mapsto y^2 - x^2$$

below



If $g : X \subset \mathbb{R}^3 \to \mathbb{R}$ is a function then its graph $\Gamma(f) \subset \mathbb{R}^4$ is the set

$$(x, y, z, w) \mid w = g(x, y, z)\}$$

Unfortunately for us, we can no longer visualise this graph as a subset of $\mathbb{R}^4$. However, we can try to visualise the level curves $g(x, y, z) = c$ as surfaces in X.