Middlebury College

Multivariable Calculus
Spring 2018
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February 28 Lecture

Textbook Reference:

- *Vector Calculus*, Colley, 4th Edition: §3.1

Parameterised Curves

Learning Objectives:

- *Understand the distinction between a path and its image curve.*
- *Learn how to compute the velocity vector of a path and its geometric interpretation.*
- *Learn how to compute the tangent line of a curve and its geometric interpretation.*

Lines as parameterised curves: Consider the following distinct parameterisations of a line L:

$$\underline{r}_1(t) = \begin{bmatrix} -1+2t \\ t \end{bmatrix}, \quad t \in \mathbb{R}, \quad \text{and} \quad \underline{r}_2(s) = \begin{bmatrix} 5-4s \\ 3-2s \end{bmatrix}, \quad s \in \mathbb{R}$$

1. Connect the mathematical object with the correct terminology

path $\qquad\qquad$ $\underline{r}_1(t)$

$\qquad\qquad\qquad$ L

curve $\qquad\qquad$ $\underline{r}_2(s)$

2. Compute the velocity vectors $\underline{r}_1'(t)$ and $\underline{r}_2'(s)$.

$\underline{r}_1'(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underline{r}_2'(s) = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

3. Given a parameteric description of a line $\underline{r}(t) = \overrightarrow{OP} + t\overrightarrow{u}$, what is the velocity $\underline{r}'(t)$? What is the speed of the path?

Velocity : $\overrightarrow{u} = \underline{r}'(t)$

Speed : $|\overrightarrow{u}|$

4. In your groups, discuss the correctness of following statement:

"the velocity vector of a line is constant"

Tangent lines of curves: Given a path $\underline{x}(t)$ with image curve C, the tangent line to C at the point $\underline{x}_0 = \underline{x}(t_0)$ is

$$\underline{l}(s) = \underline{x}_0 + s\underline{x}'(t_0), \quad s \in \mathbb{R}$$

1. Consider the path $\underline{x}(t) = te^{-t}\underline{i} + e^t\underline{j}$, $t \in \mathbb{R}$. Compute the tangent line to the image curve of $\underline{x}$ at the point $\underline{x}(0)$. Sketch the tangent line in the plane.

$$\underline{x}'(t) = (e^{-t} - te^{-t})\underline{i} + e^t\underline{j} = \begin{bmatrix} e^{-t} - te^{-t} \\ e^t \end{bmatrix}$$

$$\underline{x}'(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underline{l}(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{x}(-1) = \begin{bmatrix} -e \\ e^{-1} \end{bmatrix} \quad A$$

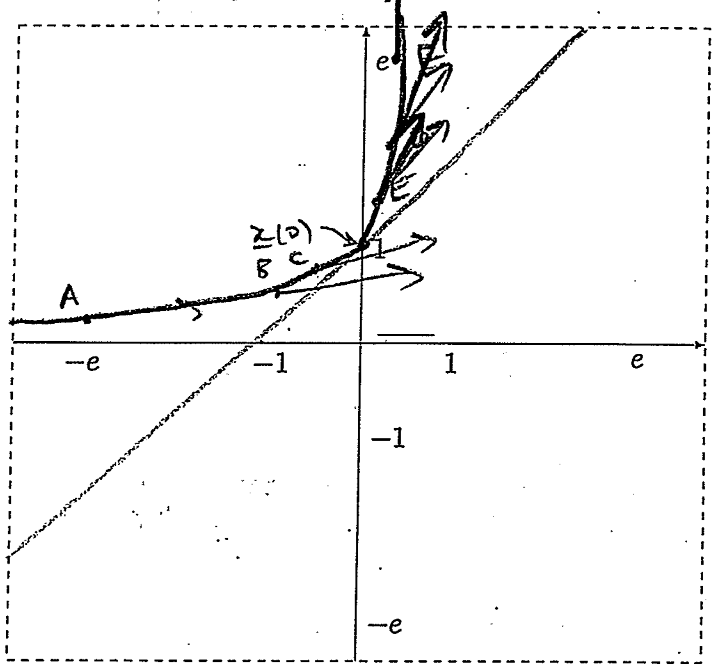
$$\underline{x}(-1/2) = \begin{bmatrix} -1/2\, e^{1/2} \\ e^{-1/2} \end{bmatrix} \quad B$$

$$\underline{x}(-1/4) = \begin{bmatrix} -1/4\, e^{1/4} \\ e^{-1/4} \end{bmatrix} \quad C$$

$$\underline{x}(1/4) = \begin{bmatrix} 1/4\, e^{-1/4} \\ e^{1/4} \end{bmatrix} \quad D$$

$$\underline{x}(1/2) = \begin{bmatrix} 1/2\, e^{-1/2} \\ e^{1/2} \end{bmatrix} \quad E$$

$$\underline{x}(1) = \begin{bmatrix} e^{-1} \\ e \end{bmatrix} \quad F$$



2. Compute the velocity vector of $\underline{x}$ at $\underline{x}(t)$, where $t = -1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1$. (*You should split up the computations among your group*)

$$\underline{x}'(-1) = \begin{bmatrix} 2e \\ e^{-1} \end{bmatrix}, \quad \underline{x}'(-1/2) = \begin{bmatrix} 3/2\, e^{1/2} \\ e^{-1/2} \end{bmatrix} \quad \underline{x}'(-1/4) = \begin{bmatrix} \frac{5}{4} e^{1/4} \\ e^{-1/4} \end{bmatrix}$$

$$\underline{x}'(1) = \begin{bmatrix} 0 \\ e \end{bmatrix} \quad \underline{x}'(1/2) = \begin{bmatrix} 1/2\, e^{-1/2} \\ e^{1/2} \end{bmatrix} \quad \underline{x}'(1/4) = \begin{bmatrix} 3/4\, e^{-1/4} \\ e^{1/4} \end{bmatrix}$$

3. Use your computations to give an approximate sketch of the image curve of $\underline{x}(t)$, for t near to 0.

Sketching parameterised curves:

Consider the path $\underline{x}(t) = \begin{bmatrix} t^2 \\ t^3 - t \end{bmatrix}$, $t \in \mathbb{R}$. Denote its image curve by C. This is the planar curve described by the equation $y^2 = x(x-1)^2$.

1. Find $t_1 \neq t_2$ so that $\underline{x}(t_1) = \underline{x}(t_2)$.(*Hint: it must be the case that* $t_1 = -t_2$) How can you interpret your solution geometrically?

If $\underline{x}(t_1) = \underline{x}(t_2)$ then

$(t_1 = -t_2)$ $\quad t_1^3 - t_1 = t_2^3 - t_2$

$\Rightarrow -t_2^3 + t_2 = t_2^3 - t_2$

$\Rightarrow 0 = 2t_2(t_2^2 - 1)$

$\Rightarrow t_2 = 0$ (then $t_1 = -t_2 = 0$, but want $t_1 \neq t_2$)

or $t_2 = \pm 1$

$\Rightarrow t_1 = 1$

$t_2 = -1$

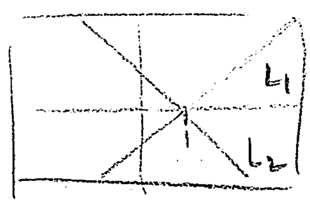
Since $\underline{x}(-1) = \underline{x}(1)$, curve must intersect itself.

2. Determine the tangent lines to C at $\underline{x}(t_1)$ and $\underline{x}(t_2)$.

$\underline{x}'(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\qquad \underline{x}'(-1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$\underline{l}_1(s) = \underline{x}(1) + s\underline{x}'(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ 2s \end{bmatrix}$

$\underline{l}_2(s) = \underline{x}(-1) + s\underline{x}'(-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

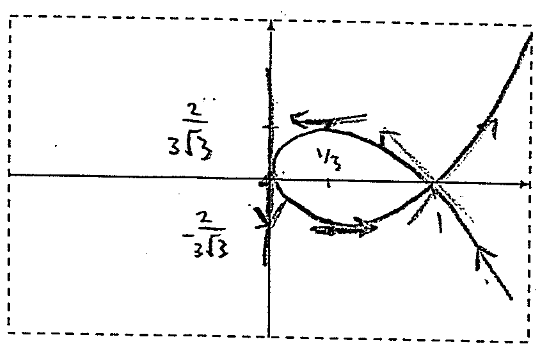


3. For which t is the tangent line to C at $\underline{x}(t)$ horizontal? Vertical?

Horizontal: When $\underline{x}'(t)$ parallel to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ie $\underline{x}'(t) = \begin{bmatrix} 2t \\ 3t^2 - 1 \end{bmatrix}$ $\Rightarrow t = \pm\frac{1}{\sqrt{3}}$

Vertical: $\underline{x}'(t)$ — " — $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ i.e. $\underline{x}'(t) = \begin{bmatrix} 2t \\ 3t^2 - 1 \end{bmatrix}$ $\Rightarrow t = 0$

4. Using what you've found above, sketch C below.



Vertical tangent line: $t = 0$

$\Rightarrow \underline{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Horizontal: $t = \pm\frac{1}{\sqrt{3}}$

$\underline{x}\left(\frac{1}{\sqrt{3}}\right) = \begin{bmatrix} 1/3 \\ \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3\sqrt{3} \end{bmatrix}$

$\underline{x}(-1/\sqrt{3}) = \begin{bmatrix} 1/3 \\ 2/3\sqrt{3} \end{bmatrix}$