Middlebury College

Multivariable Calculus
Spring 2018
Contact: `gmelvin@middlebury.edu`

February 28 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §3.1

Parameterised Curves

Learning Objectives:
- *Understand the distinction between a path and its image curve.*
- *Learn how to compute the velocity vector of a path and its geometric interpretation.*
- *Learn how to compute the tangent line of a curve and its geometric interpretation.*

Lines as parameterised curves: Consider the following distinct parameterisations of a line L:

$$\underline{r}_1(t) = \begin{bmatrix} -1+2t \\ t \end{bmatrix}, \quad t \in \mathbb{R}, \quad \text{and} \quad \underline{r}_2(s) = \begin{bmatrix} 5-4s \\ 3-2s \end{bmatrix}, \quad s \in \mathbb{R}$$

1. Connect the mathematical object with the correct terminology

	$\underline{r}_1(t)$
path	
	L
curve	
	$\underline{r}_2(s)$

2. Compute the velocity vectors $\underline{r}_1'(t)$ and $\underline{r}_2'(s)$.

3. Given a parameteric description of a line $\underline{r}(t) = \overrightarrow{OP} + t\,\overrightarrow{u}$, what is the velocity $\underline{r}'(t)$? What is the speed of the path?

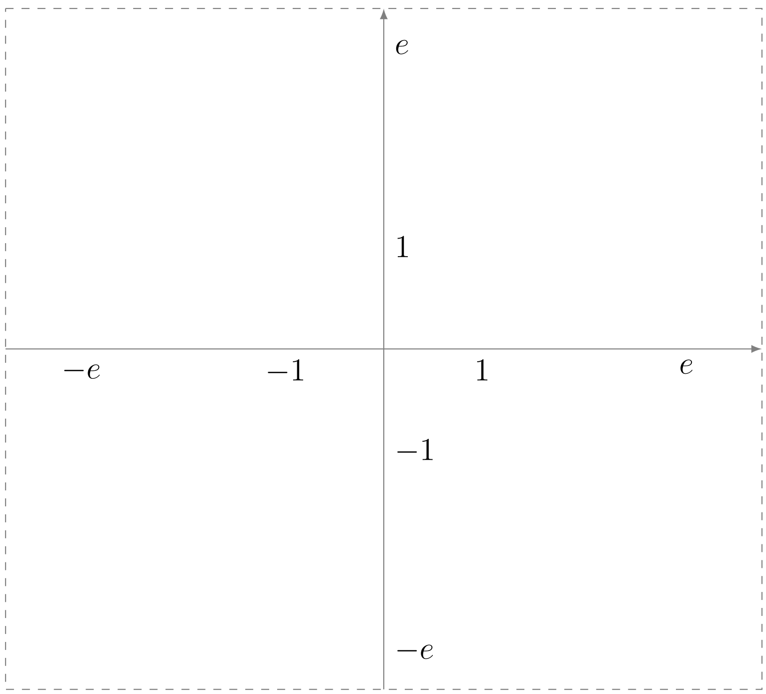
4. In your groups, discuss the correctness of following statement:

"the velocity vector of a line is constant"

Tangent lines of curves: Given a path $\underline{x}(t)$ with image curve C, the tangent line to C at the point $\underline{x}_0 = \underline{x}(t_0)$ is

$$\underline{l}(s) = \underline{x}_0 + s\underline{x}'(t_0), \quad s \in \mathbb{R}$$

1. Consider the path $\underline{x}(t) = te^{-t}\underline{i} + e^t\underline{j}$, $t \in \mathbb{R}$. Compute the tangent line to the image curve of $\underline{x}$ at the point $\underline{x}(0)$. Sketch the tangent line in the plane.



2. Compute the velocity vector of $\underline{x}$ at $\underline{x}(t)$, where $t = -1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1$. (*You should split up the computations among your group*)

3. Use your computations to give an approximate sketch of the image curve of $\underline{x}(t)$, for t near to 0.

Sketching parameterised curves:

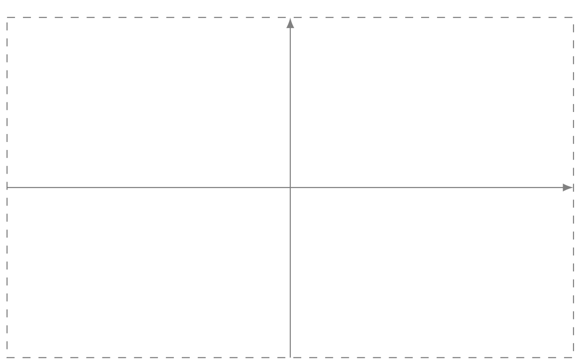
Consider the path $\underline{x}(t) = \begin{bmatrix} t^2 \\ t^3 - t \end{bmatrix}$, $t \in \mathbb{R}$. Denote its image curve by C. This is the planar curve described by the equation $y^2 = x(x-1)^2$.

1. Find $t_1 \neq t_2$ so that $\underline{x}(t_1) = \underline{x}(t_2)$.(*Hint: it must be the case that* $t_1 = -t_2$) How can you interpret your solution geometrically?

2. Determine the tangent lines to C at $\underline{x}(t_1)$ and $\underline{x}(t_2)$.

3. For which t is the tangent line to C at $\underline{x}(t)$ horizontal? Vertical?

4. Using what you've found above, sketch C below.



Misc:

1. **Tangent Slopes**

   Let $\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a path in $\mathbb{R}^2$.

   (a) Determine conditions for the tangent line to the image curve of $\underline{x}$ to be horizontal/vertical.

   (b) Write down an expression for the slope of a tangent line to the image curve of $\underline{x}$.

   (c) Suppose that part of the image curve can be realised as a graph $y = F(x)$, for some differentiable function F. Explain why, if $\frac{dx}{dt} \neq 0$ then

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

   (d) Using implicit differentiation check that this formula holds for the parameterisation of the curve $y^2 = x(x-1)^2$ given above.

   (e) Explain why, if there are values $t = t_0$ such that $\frac{dx}{dt}(t_0$, then the image curve of $\underline{x}$ can't be realised as graph $y = f(x)$.