Multivariable Calculus
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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §1.7

Coordinate systems

Learning Objectives:
- *Gain familiarity with spherical coordinates.*
- *Be able to translate between Cartesian/cylindrical/spherical coordinate systems.*
- *Gain familiarity describing geometric objects in new coordinate systems.*

Keywords: *cylindrical coordinates, spherical coordinates.*

In this lecture we will describe some new coordinate systems in $\mathbb{R}^3$.

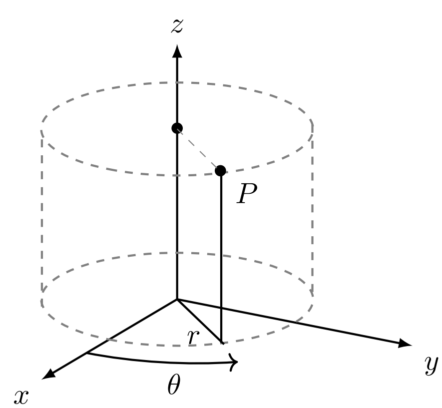
Coordinates in space

The Cartesian coordinates in the plane can be extended to space: we add in a new z coordinate, where z measures units distance in the direction $\underline{k} \stackrel{def}{=} \underline{i} \times \underline{j}$. As in the $\mathbb{R}^2$ case, we could also describe points in space (once we've fixed an origin O) by giving three linearly independent vectors $\underline{u}, \underline{v}, \underline{w}$ and determining a new coordinate system with respect to the resulting basis.

Cylindrical coordinates: Polar coordinates provide us with a coordinate system in the plane and we can extend this to a coordinate system in $\mathbb{R}^3$.

Given a point P in space, use polar coordinates to describe the projection of P onto the xy-plane: denote this projection (r, θ). Then, P can be described by the triple (r, θ, z). We say that (r, θ, z) obtained in this way are the **cylindrical coordinates** of P.

The terminology is justified by considering the following diagram:



Cartesian $\leftrightarrow$ cylindrical coordinate transformation

$$\text{Cylindrical to Cartesian:} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \qquad (1)$$

$$\text{Cartesian to cylindrical:} \qquad \begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \tfrac{y}{x} \\ z &= z \end{aligned} \qquad (2)$$

Remark:

1. As with polar coordinates, all points in $\mathbb{R}^3$ except for the z-axis have a unique set of cylindrical coordinates. Any point $(0, 0, c)$ on the z-axis has cylindrical coordinates $(0, \theta, c)$, where θ can be any angle.

2. Cylindrical coordinates are useful when studying objects possessing rotational symmetry (about the z-axis).

Example:

1. The surface in $\mathbb{R}^3$ described by $r = c$ is the cylinder, centred at the origin, parallel to the z-axis, and having radius c. In Cartesian coordinates, we see that a cylinder (parallel to the z-axis) is therefore given by the equation

   $$\sqrt{x^2 + y^2} = c \quad \text{or, equivalently} \quad x^2 + y^2 = c^2.$$

   This example highlights an important point: *if an equation does not contain a coordinate, then the resulting object described by the equation extends infinitely in both directions parallel to the axis of the missing coordinate.*

2. The surface in $\mathbb{R}^3$ described by the equation $\tan\theta = m$, is the plane containing the z-axis and the line $y = mx$.

3. The surface in $\mathbb{R}^3$ described by the equation $z^2 + r^2 = 400$, $r \in \mathbb{R}$, is a sphere of radius 20 centred at the origin: in Cartesian coordinates the equation becomes

   $$z^2 + r^2 = 400 \quad \Longrightarrow \quad z^2 + x^2 + y^2 = 20^2$$

   If (x, y, z) lies on the surface described by this equation then it must be at distance 20 from the origin. All points in space at a fixed distance from the origin define a sphere centred at the origin.

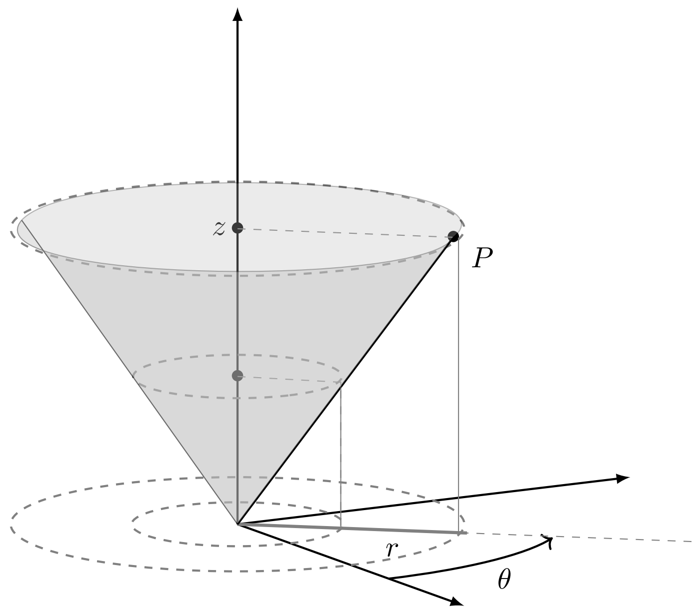
4. The surface in described by the equation $z = r$, $r \in \mathbb{R}$, is a *double-napped cone*: points on the surface $r = z$ intersecting the $z = c$ plane lie above/below the circle having radius c centred at the origin. If we restrict r to be nonnegative then we obtain the top half (**nappe**) of the cone.

   In particular, this allows us to determine an equation in Cartesian coordinates for the double-napped cone:

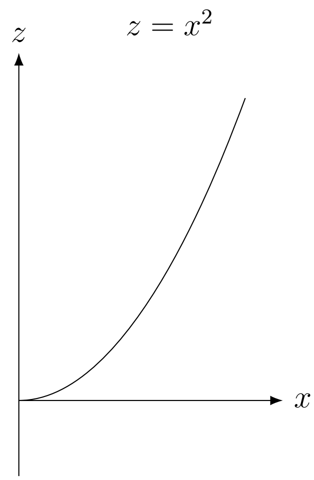
   $$z = r \quad \Longrightarrow \quad z^2 = r^2 = x^2 + y^2$$

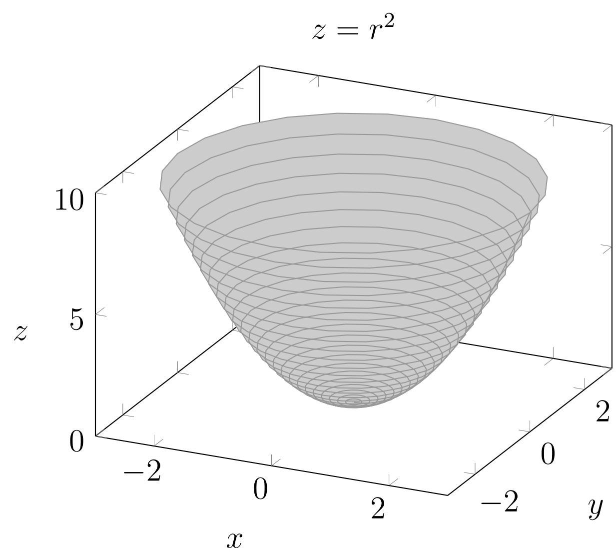
Double-napped cone:

- $z^2 = x^2 + y^2$ (Cartesian)
- $z = r$ (Cylindrical)



5. **[Important Example:]** More generally, if $z = f(r)$ then the surface described by this equation is the surface of revolution obtained by rotating the graph $z = f(x)$ (in the xz-plane) about the z-axis. For example, the surface $z = r^2$ is a **paraboloid** about the z-axis:

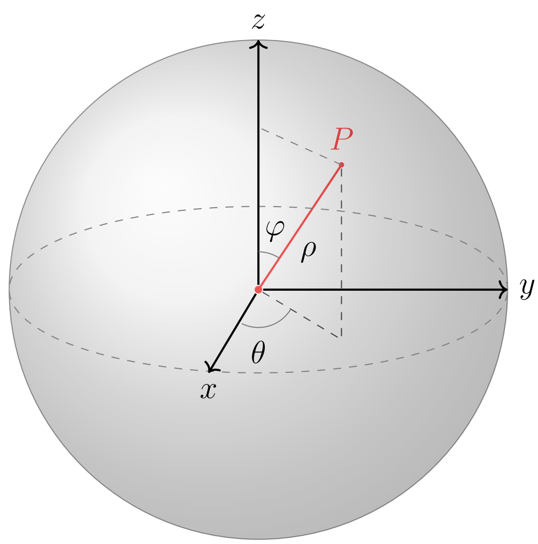




Paraboloid:

- $z = x^2 + y^2$ (Cartesian)
- $z = r^2$ (Cylindrical)

Spherical Coordinates: In analogy to polar coordinates, we can describe a point P in space by specifying the unique sphere of radius ρ (centred at the origin) that P lies on and determining *latitudinal* and *longitudinal* coordinates, (φ, θ).



So as to not confuse the radius of the sphere with the radius of a circle, we use the symbol ρ (pronounced *rho*). We call (ρ, φ, θ) the **spherical coordinates** of P.

We impose the following restrictions:

$$\rho \geq 0, \quad 0 \leq \varphi \leq \pi, \quad 0 \leq \theta < 2\pi.$$

With these restrictions, every point in space, except for points on the z-axis, have a uniquely determined set of spherical coordinates.

Coordinate transformations in space

$$\text{Spherical/cylindrical:} \qquad \begin{aligned} r &= \rho \sin\varphi \\ \theta &= \theta \\ z &= \rho\cos\varphi \end{aligned} \qquad \begin{aligned} \rho^2 &= r^2 + z^2 \\ \theta &= \theta \\ \tan\varphi &= r/z \end{aligned} \tag{3}$$

$$\text{Spherical/Cartesian:} \qquad \begin{aligned} x &= \rho \sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned} \qquad \begin{aligned} \rho^2 &= x^2 + y^2 + z^2 \\ \tan\theta &= \tfrac{y}{x} \\ \tan\varphi &= \tfrac{\sqrt{x^2+y^2}}{z} \end{aligned} \tag{4}$$

Example:

1. Using the above change-of-coordinate formula, we see that the point $P = (0, \sqrt{3}, 1)$ (in Cartesian coordinates) has spherical coordinates $(\rho, \varphi, \theta) = (2, \pi/3, \pi/2)$.

2. The surface defined by the equation $\rho = c$ is a sphere of radius c centred at the origin.

3. The surface defined by the equation $\varphi = \pi/4$ is a single-napped cone: using (4) above gives $z = \sqrt{x^2 + y^2}$. Note: since $\tan\varphi = 1 > 0$, we must choose $z > 0$.

4. The surface defined by the equation $\rho = 6\cos\varphi$ defines a sphere of radius 3 centred at $(0,0,3)$: multiplying both sides of the equation by ρ gives

$$\rho^2 = 6\rho\cos\varphi \qquad \Longrightarrow \qquad x^2 + y^2 + z^2 = 6z$$

Rearranging and completing the square we find

$$x^2 + y^2 + (z-3)^2 = 3^2.$$

Remark: the equation of a sphere of radius a centred at $P = (x_0, y_0, z_0)$ is given by the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Indeed, this sphere consists of all points $Q = (x, y, z)$ satisfying $|\overrightarrow{PQ}| = a$. Squaring both sides gives

$$|\overrightarrow{PQ}|^2 = a^2 \qquad \Longrightarrow \qquad \overrightarrow{PQ} \cdot \overrightarrow{PQ} = a^2 \qquad \Longrightarrow \qquad \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = a^2$$

and the claim follows.

Sphere of radius a centred at (x_0, y_0, z_0):

$$a^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$