Middlebury College

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February 21 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §1.7

Coordinate systems

Learning Objectives:
- *Gain familiarity with polar coordinates.*
- *Be able to graph basic polar curves.*
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- *Gain familiarity with cylindrical coordinates.*

In this lecture we will describe some new coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$.

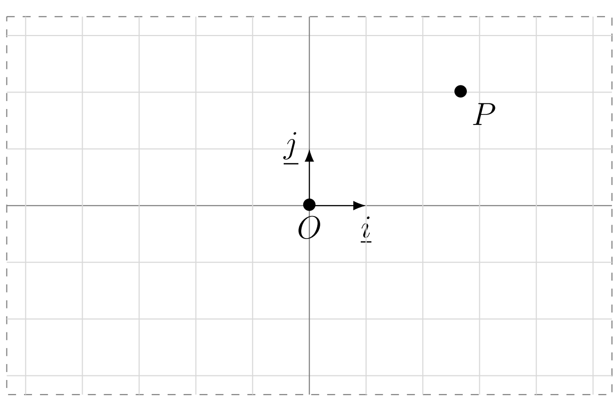
Coordinates in the plane

Consider the plane $\mathbb{R}^2$ - this is a flat two-dimensional surface that is infinite in all directions. The basic question is

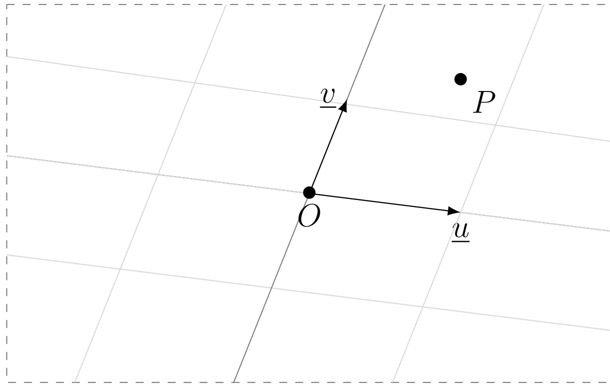
Question: how can we describe points in the plane?

To the Greeks a point just *was*: we would care about describing points when they appeared in a problem of geometry and were a (un)known distance from another point.

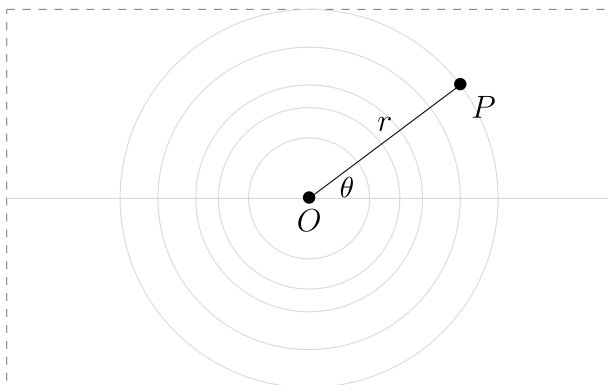
Many centuries later, **Descartes** (and, independently, **Fermat**) came up with the following revolutionary idea: fix a point in the plane (call it O), choose two perpendicular fundamental directions (let's call them $\underline{i}$ and $\underline{j}$) and basic units of length, describe points relative to these fundamental directions. This, of course, leads to our usual **Cartesian** (or **rectangular**) description of the plane using (x, y) coordinates.



In linear algebra terms, the vectors $\underline{i}, \underline{j}$ are linearly independent and therefore provide a basis of $\mathbb{R}^2$. We could extend this idea by choosing any two linearly independent vectors $\underline{u}, \underline{v}$ to determine a coordinate system on $\mathbb{R}^2$:



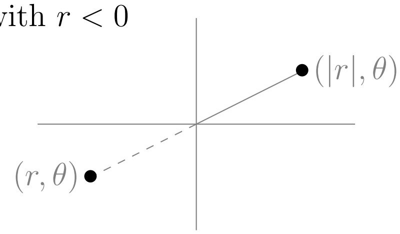
Polar coordinates: a useful coordinate system in the plane, called the **polar coordinate system**, is defined as follows: fix an origin O. Any point P (distinct from the origin O) lies on a unique circle of some radius r. To determine precisely where the point P is on the circle, we fix a line through the origin (which we assume is horizontal) and measure (counterclockwise) the angle θ subtended by P from this line.



The point P is represented by the pair (r, θ), the **polar coordinates** of P. To remove ambiguity, **always choose** $0 \leq \theta < 2\pi$.

Convention: Sometimes we will also allow r to take negative values, to be interpreted as follows: given polar coordinates (r, θ), with $r < 0$, consider the ray making angle θ with the x-axis, and instead of moving $|r|$ units away from the origin along this ray, go $|r|$ units in the *opposite* direction.

Interpreting (r, θ) with $r < 0$



Remark: Restricting $0 \leq \theta < 2\pi$, $r \geq 0$, ensures that any point in the plane, apart from the origin O, has a unique set of polar coordinates.

Example:

1. The point $P = (2,2)$ (in Cartesian coordinates) lies on a circle of radius $\sqrt{2^2+2^2} = 2\sqrt{2}$, and we have $\tan\theta = 1$. Hence, since $x, y > 0$, we must have $\theta = \frac{\pi}{4}$. Therefore, in polar coordinates the point P is represented by $(r,\theta) = (2\sqrt{2}, \theta)$.

2. Consider the point P which is represented by $(5, \pi/6)$ in polar coordinates. Then, P lies in the first quadrant on the arc of the circle, centred at O, of radius 5. Recalling some basic trigonometry we have, in Cartesian coordinates, $P = (x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$ i.e. $P = (5\sqrt{3}/2, 5/2)$.

3. The origin is **weird**: it is given, in polar coordinates, by $(0,\theta)$, for *any* θ.

Since a point P in the plane doesn't care about how we represent it (it just *is*, as the Greeks would say), we should be able to change between polar and Cartesian coordinate representations for P (analogous to change-of-coordinate transformations in linear algebra).

Cartesian ↔ polar coordinate transformation

Polar to Cartesian:	$x = r\cos\theta$ $y = r\sin\theta$	(1)
Cartesian to polar:	$r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$	(2)

Caution: the Cartesian to polar change-of-coordinate formula in (2) do not specify (r,θ) uniquely in terms of x, y. Read through p.64 of the textbook for discussion.

Polar equations: we can describe geometric objects in the plane using equations in polar coordinates.

1. In polar coordinates (r,θ), a circle having radius c, centred at the origin, is defined by the equation $r = c$.

2. The straight line through the origin with slope m is given by the equation $\tan\theta = m$.

3. The vertical line through $(0,1)$ is given by $r = \sec\theta$: rearranging this equation gives $1 = r\cos\theta = x$.

4. The equation $r = 2\cos\theta$ describes a circle of radius 1 centred at $(1,0)$: multiplying both sides by r gives

$$r^2 = 2r\cos\theta \quad \Longrightarrow \quad x^2 + y^2 = 2x.$$

   Completing the square gives

$$(x-1)^2 + y^2 = 1.$$

Remark: determining the shapes described by a polar equation is tricky and takes some getting used to. Can you see what shape is described by the polar equation $r = \theta$?

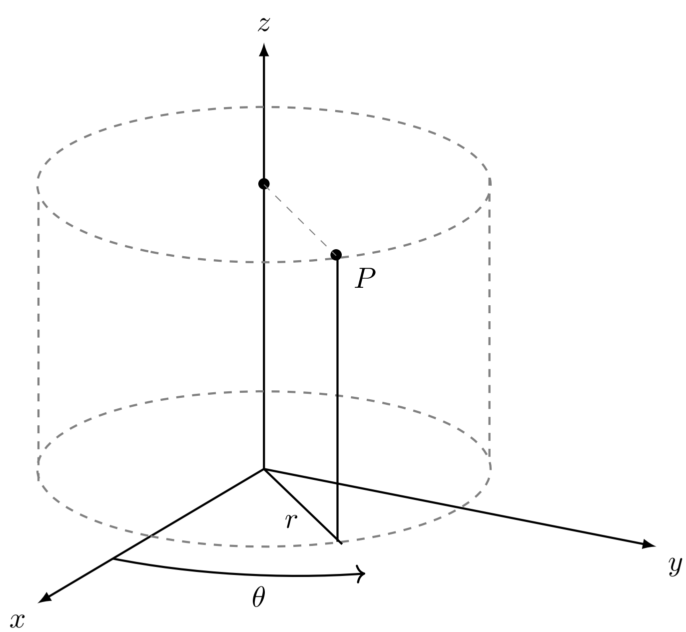
Coordinates in space

The Cartesian coordinates in the plane can be extended to space: we add in a new z coordinate, where z measures units distance in the direction $\underline{k} \stackrel{def}{=} \underline{i} \times \underline{j}$. As in the $\mathbb{R}^2$ case, we could also describe points in space (once we've fixed an origin O) by giving three linearly independent vectors $\underline{u}, \underline{v}, \underline{w}$ and determining a new coordinate system with respect to the resulting basis.

Cylindrical coordinates: Polar coordinates provide us with a coordinate system in the plane and we can extend this to a coordinate system in $\mathbb{R}^3$.

Given a point P in space, use polar coordinates to describe the projection of P onto the xy-plane: denote this projection (r, θ). Then, P can be described by the triple (r, θ, z). We say that (r, θ, z) obtained in this way are the **cylindrical coordinates** of P.

The terminology is justified by considering the following diagram:



Cartesian ↔ cylindrical coordinate transformation

Cylindrical to Cartesian:
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \tag{1}$$

Cartesian to cylindrical:
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z \tag{2}$$

Remark:

1. As with polar coordinates, all points in $\mathbb{R}^3$ except for the z-axis have a unique set of cylindrical coordinates. Any point $(0, 0, c)$ on the z-axis has cylindrical coordinates $(0, \theta, c)$, where θ can be any angle.

2. Cylindrical coordinates are useful when studying objects possessing rotational symmetry (about the z-axis).

Example:

1. The surface in $\mathbb{R}^3$ described by $r = c$ is the cylinder, centred at the origin, parallel to the z-axis, and having radius c. In Cartesian coordinates, we see that a cylinder (parallel to the z-axis) is therefore given by the equation

   $$\sqrt{x^2 + y^2} = c \quad \text{or, equivalently} \quad x^2 + y^2 = c^2.$$

   This example highlights an important point: *if an equation does not contain a coordinate, then the resulting object described by the equation extends infinitely in both directions parallel to the axis of the missing coordinate.*

2. The surface in $\mathbb{R}^3$ described by the equation $\tan\theta = m$, is the plane containing the z-axis and the line $y = mx$.

3. The surface in $\mathbb{R}^3$ described by the equation $z^2 + r^2 = 400$, $r \in \mathbb{R}$, is a sphere of radius 20 centred at the origin: in Cartesian coordinates the equation becomes

   $$z^2 + r^2 = 400 \qquad \Longrightarrow \qquad z^2 + x^2 + y^2 = 20^2$$

   If (x, y, z) lies on the surface described by this equation then it must be at distance 20 from the origin. All points in space at a fixed distance from the origin define a sphere centred at the origin.