Middlebury College

Multivariable Calculus
Spring 2018
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February 19 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §1.5

Affine Geometry: lines & planes

Learning Objectives:
- *Understand how to determine a parametric description of a line.*
- *Understand how to determine the equation of a plane.*
- *Understand the difference between a parametric description and an equation description of a geometric object.*

Parametric descriptions of lines: given a point P and direction vector $\vec{u}$ we can give a parametric description of the line through P parallel to $\vec{u}$:

$$\underline{r}(t) = \overrightarrow{OP} + t\vec{u}, \quad t \in \mathbb{R}.$$

1. Let $P = (1,1,1)$, $Q = (2,0,1)$. Give the parametric description of the line L passing through P and Q.

$$\vec{u} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix}2\\0\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\underline{r}(t) = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}1+t\\1-t\\1\end{bmatrix} \quad \text{i.e.} \quad \begin{matrix}x_1 = 1+t\\x_2 = 1-t\\x_3 = 1\end{matrix}$$

2. Consider the line with parametric description

$$\underline{r}(t) = \begin{bmatrix}5t\\2-5t\\1\end{bmatrix}, \quad t \in \mathbb{R} \quad = \begin{bmatrix}5\\2\\1\end{bmatrix} + t\begin{bmatrix}5\\-5\\0\end{bmatrix}$$

Explain why this line is equal to L. (**Parametric descriptions are non-unique**)

Sufficient to show:

- $(0,2,1) \in L$ i.e. $\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + t_0\begin{bmatrix}1\\-1\\0\end{bmatrix}$, some $t_0 \in \mathbb{R}$
- $\begin{bmatrix}5\\-5\\0\end{bmatrix}$ parallel to $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ (obvs)

Solve for t: $0 = 1+t$, $2 = 1-t$, $1 = 1$ ✓

$t = -1$, consistent

3. Does the point $R = (2,-1,2)$ lie on L? How can you check?

To check: can we find t_0 s.t.

$$\begin{bmatrix}2\\-1\\2\end{bmatrix} = \begin{bmatrix}1+t_0\\1-t_0\\1\end{bmatrix} \quad \text{i.e.} \quad \begin{matrix}2 = 1+t_0\\-1 = 1-t_0\\2 = 1 \leftarrow \times\end{matrix}$$

No. $R \notin L$.

Equations of planes: Let P be a point, $\vec{n}$ a direction vector. There is a unique plane containing P and perpendicular to $\vec{n}$ (**a normal vector**). It is defined by the equation

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\overrightarrow{OP}, \qquad \text{where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1. Determine the equation of the plane containing the points $P=(1,0,0)$, $Q=(0,1,0)$, $R=(0,0,1)$.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\vec{n}\cdot\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + x_2 + x_3$$

$$\vec{n}\cdot\overrightarrow{OP} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\boxed{x_1 + x_2 + x_3 = 1}$$

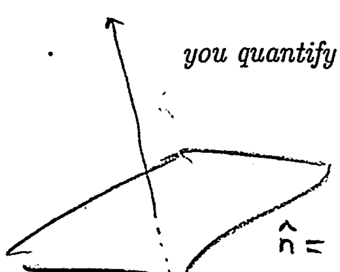
2. What is the equation of the plane containing the point $(2,5,1)$ and perpendicular to the line L from the first set of problems above?

Direction vector for L can be taken for normal to plane.

$$\Rightarrow \vec{n} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \qquad \boxed{x_1 - x_2 = -3}$$

$$\Rightarrow \vec{n}\cdot\vec{x} = x_1 - x_2, \quad \vec{n}\cdot\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = -3.$$

3. Determine whether the plane defined by the equation $2x+5z=10$ and the line $\vec{r}(t) = \begin{bmatrix} 1 \\ 2+t \\ 2t \end{bmatrix}$, $t\in\mathbb{R}$, are perpendicular, parallel, or neither. (*How can you quantify these possibilities using a normal vector?*)



$$\hat{n} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$2x+5z=10$

$$r(t) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

perpendicular : $\vec{n} \parallel \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ ✗

parallel : $\vec{n} \perp \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ ✗

ie $\vec{n}\cdot\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 10 \neq 0$

$\boxed{\text{neither}}$

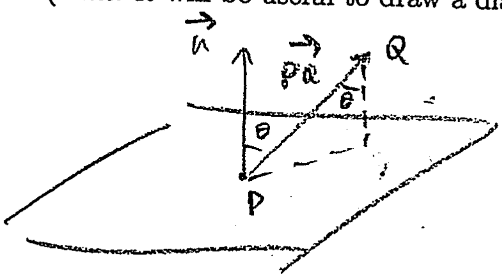
this means $(0,1,2)$ is on the plane.

Another distance formulae:

1. Let Π be the plane with normal vector $\vec{n}$ passing through P. Let Q be any point. Explain why the distance from Q to Π is

$$\frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$$

(*Hint:* it will be useful to draw a diagram representing the problem)

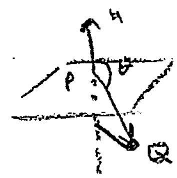


distance $= |\vec{PQ}|\cos\theta$

Use cosine formula for dot product

$$= \frac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$$

we put absolute value bars in case Q is on 'other side' of Π' to $\vec{n}$ i.e.



2. Let Π be the plane with normal $\vec{n} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ passing through $P = (1,0,0)$. Let $Q = (-1,2,2)$.

(a) Explain why Q is not on the plane Π.

Equation of plane: $x_1 + 2x_2 + x_3 = \vec{n}\cdot\vec{OP} = 1$

coordinates of Q don't satisfy equation, (ie $(-1)\cdot 1 + 2\cdot 2 + 2\cdot 1 = 5 \neq 1$)

(b) Determine the point of intersection between Π and the line through Q in the direction $\vec{n}$

Parametric description of line

$$r(t) = \vec{OQ} + t\vec{n}$$

$$= \begin{bmatrix}-1\\2\\2\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-1+t\\2+2t\\2+t\end{bmatrix}$$

r(t) intersects line when its coordinates satisfy the equation i.e.

$$1 = -1+t \quad 2\cdot(2+2t) + (2+t)$$

$$= 5+6t \Rightarrow t = -\frac{2}{3}$$

ie $r(-\frac{2}{3}) = \begin{bmatrix}-5/3\\2/3\\4/3\end{bmatrix} \Rightarrow (-5/3, 2/3, 4/3)$ is point of intersection

Misc:

1. Let Π and Π' be non-parallel planes, and let $\vec{n}$, $\vec{n}'$ be corresponding normal vectors. In your groups, give a geometric explanation for why the line of intersection $\Pi \cap \Pi'$ is parallel to $\vec{n} \times \vec{n}'$. If possible, use the space below to provide a diagram supporting your explanation.

Line of intersection L is contained in both Π and Π'

$\Rightarrow$ L perpendicular to $\vec{n}$ and $\vec{n}'$

$\Rightarrow$ L parallel to $\vec{n} \times \vec{n}'$

2) Consider $ax_1 + bx_2 + cx_3 = 0 \; : \Pi$

$px_1 + qx_2 + rx_3 = 0 \; : \Pi'$

(full rank).

Then, solution set are vectors satisfying both equations

$\Leftrightarrow$ solution set consists of vectors on both planes Π and Π'

$\Leftrightarrow$ solution set $= \Pi \cap \Pi'$

Hence, solution set is line parallel to $\vec{n} \times \vec{n}' = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ (passing through $\underline{0}$)

ie subspace spanned by