Middlebury College

Multivariable Calculus
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Contact: `gmelvin@middlebury.edu`

February 16 Lecture

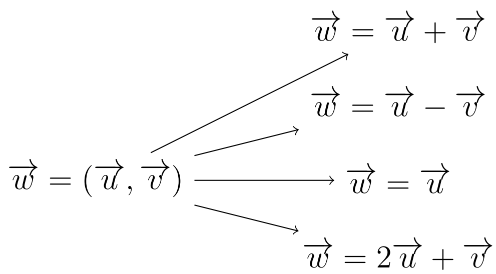
Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: § 1.4

The Cross Product

Important note: Today we restrict our attention to $\mathbb{R}^3$ (i.e. 3-space).

> **Question:** given two vectors in space, $\vec{u}, \vec{v}$, how can we construct a third vector $\vec{w}$?

There are many ways; for example, given $\vec{u}, \vec{v}$ we could associated the following vectors



There are many ways to answer the question; no rhyme nor reason to it.

Consider the following geometric construction: let $\vec{u}, \vec{v}$ be displacement vectors in space. Assume these vectors have the same starting point P. to construct a new arrow $\vec{w}$ we require

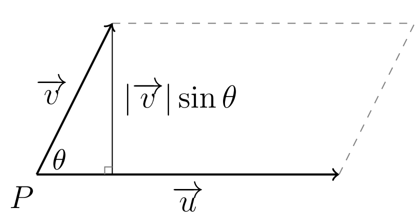
(A) magnitude,

(B) direction.

(A) Magnitude: we declare that we want

$$|\vec{w}| = \text{area of parallelogram spanned by } \vec{u}, \vec{v} = \underbrace{|\vec{u}|}_{\text{base}}\underbrace{|\vec{v}|\sin\theta}_{\text{height}}$$

Here we measure θ so that $0 \le \theta \le \pi$.



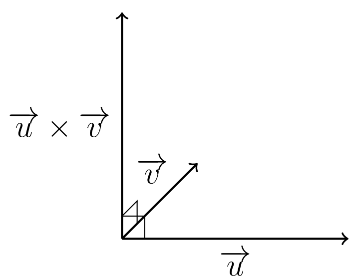
Remark:

1. Observe: if $\overrightarrow{u} \perp \overrightarrow{v}$ and $|\overrightarrow{u}| = |\overrightarrow{v}| = 1$ then $|\overrightarrow{w}| = 1$.
2. If $\overrightarrow{u} = \overrightarrow{0}$, $\overrightarrow{v} = \overrightarrow{0}$, or $\overrightarrow{u} || \overrightarrow{v}$ then

$$|\overrightarrow{w}| = 0 \implies \overrightarrow{w} = \overrightarrow{0}.$$

(B) Direction: By the Remark, we can assume that both $\overrightarrow{u}$ and $\overrightarrow{v}$ are nonzero and non-parallel. Then, there is a unique line passing through P that's perpendicular to the plane spanned by $\overrightarrow{u}$ and $\overrightarrow{v}$. We choose $\overrightarrow{w}$ to be parallel to this line (starting at P), and assign it's direction using the **right hand rule**. Thus, $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ forms a **right-handed triad**.

We **define** $\overrightarrow{u} \times \overrightarrow{v}$, the **cross product of** $\overrightarrow{u}$ **and** $\overrightarrow{v}$, to be the vector just defined.



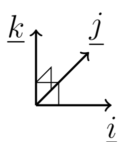
In particular, by construction

$$|\overrightarrow{u} \times \overrightarrow{v}| = |\overrightarrow{u}||\overrightarrow{v}| \sin\theta$$

It's important to remember that we take the angle θ between $\overrightarrow{u}$ and $\overrightarrow{v}$ so that $0 \le \theta \le \pi$ (why?).

Example:

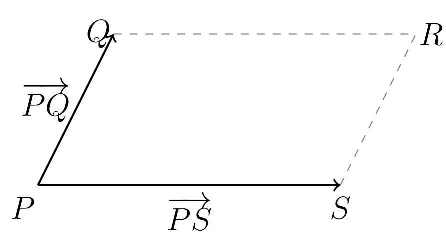
1. Denote the standard basis vectors in $\mathbb{R}^3$ by $\underline{i}, \underline{j}, \underline{k}$.



Then,

$$\underline{i} \times \underline{j} = \underline{k}, \qquad \underline{j} \times \underline{i} = -\underline{k}$$
$$\underline{j} \times \underline{k} = \underline{i}, \qquad \underline{k} \times \underline{j} = -\underline{i}$$
$$\underline{k} \times \underline{i} = \underline{j}, \qquad \underline{i} \times \underline{k} = -\underline{j}.$$

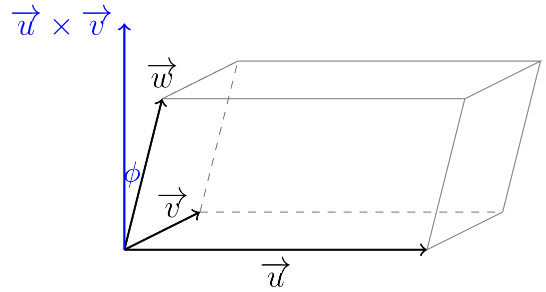
2. Let P, Q, S be points in space.



Then,

$$\text{area of } PQS = \frac{1}{2}\text{area of } PQRS = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PS}|$$

3. Consider the parallelepiped V spanned by $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$



Then,

$$\text{volume of } V = \text{base} \cdot \text{height} = |\overrightarrow{u} \times \overrightarrow{v}| \cos\phi$$

where ϕ is the angle between $\overrightarrow{w}$ and $\overrightarrow{u} \times \overrightarrow{v}$. Using the dot product to determine $\cos\phi$, we obtain

$$\text{volume of } V = \overrightarrow{w} \cdot (\overrightarrow{u} \times \overrightarrow{v}) = (\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}$$

Exercise: show that

$$(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w} = (\overrightarrow{v} \times \overrightarrow{w}) \cdot \overrightarrow{u} = (\overrightarrow{w} \times \overrightarrow{u}) \cdot \overrightarrow{v} \qquad (*)$$

(*Hint:* each of these quantities computes the volume of V)

We will now see how we can use these results to obtain some interesting properties of the cross product. First, we recall the following fact (an exercise if you've never seen it before):

Fact: Suppose $\underline{x} \cdot \underline{y} = 0$, for every $\underline{x} \in \mathbb{R}^3$. Then, $\underline{y} = \underline{0} \in \mathbb{R}^3$.

Now let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}, \overrightarrow{x}$ be vectors in $\mathbb{R}^3$. Assume that they all have same starting point (i.e. they are all position vectors relative to this common starting point). Then,

$$\begin{aligned}(\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w})) \cdot \overrightarrow{x} &\overset{(*)}{=} \overrightarrow{x} \times \overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) \\ &= (\overrightarrow{x} \times \overrightarrow{u}) \cdot \overrightarrow{v} + (\overrightarrow{x} \times \overrightarrow{u}) \cdot \overrightarrow{w} \\ &\overset{(*)}{=} (\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{x} + (\overrightarrow{u} \times \overrightarrow{w}) \cdot \overrightarrow{x} \\ &= (\overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}) \cdot \overrightarrow{x}\end{aligned}$$

Hence, bringing everything to one side, we obtain

$$(\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) - \overrightarrow{u} \times \overrightarrow{v} - \overrightarrow{u} \times \overrightarrow{w}) \cdot \overrightarrow{x} = \overrightarrow{0}$$

This is true for all $\overrightarrow{x}$, so we must have

$$\begin{aligned}&\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) - \overrightarrow{u} \times \overrightarrow{v} - \overrightarrow{u} \times \overrightarrow{w} = \overrightarrow{0} \\ \Longrightarrow \quad &\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}\end{aligned}$$

SImilarly, it can be shown that

$$(\overrightarrow{u} + \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{w}$$

Properties of the cross product
Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be vectors in space.

1. $\overrightarrow{u} \times \overrightarrow{v} = -\overrightarrow{v} \times \overrightarrow{u}$ (exercise! use the right hand rule)
2. $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}$
3. $(\overrightarrow{u} + \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{w}$
4. $(\lambda\overrightarrow{u}) \times \overrightarrow{v} = \lambda(\overrightarrow{u} \times \overrightarrow{v}) = \overrightarrow{u} \times (\lambda\overrightarrow{v})$, for any scalar λ.

What we have obtained allows us to determine a formula for the cross product as follows: take two column vectors

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

We identify these column vectors with the corresponding position vectors (i.e. the displacement vectors from the the origin to the points defined by the column vector).

Hence,

$$\underline{x} = x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$$
$$\underline{y} = y_1\underline{i} + y_2\underline{j} + y_3\underline{k}$$

Thus, using properties 2, 3, 4 above, and our determination of the cross product of the standard basis vectors, we find (after substantial rearrangement)

$$\begin{aligned} \underline{x} \times \underline{y} &= \left(x_1\underline{i} + x_2\underline{j} + x_3\underline{k}\right) \times \left(y_1\underline{i} + y_2\underline{j} + y_3\underline{k}\right) \\ &= (x_2y_3 - x_3y_2)\underline{i} - (x_1y_3 - x_3y_1)\underline{j} + (x_1y_2 - x_2y_1)\underline{k} \end{aligned}$$

See the textbook, §1.4, for further details.

Example:

1. Let $\underline{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Then,

$$\underline{x} \times \underline{y} = (1.2 - 0.(-1))\underline{i} - (2.2 - (-1).1)\underline{j} + (2.0 - 1.1)\underline{k} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$$

2. Let $P = (1, 2, 0), Q = (2, 0, 0), R = (-1, -1, -1)$. The area of the triangle PQR can be determine as follows. First, we find the displacement vectors

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$$

Then, the area of the triangle PQR is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

We compute

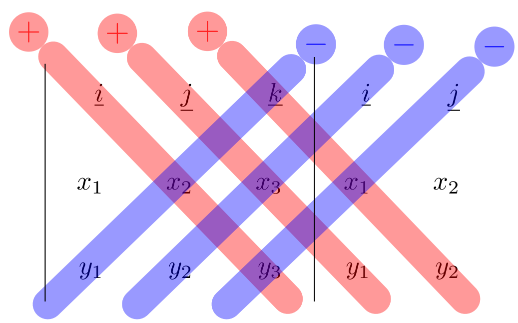
$$\overrightarrow{PQ} \times \overrightarrow{PR} = ((-2).(-1) - 0.(-3))\underline{i} - (1.(-1) - 0.(-2))\underline{j} + (1.(-3) - (-2).(-2))\underline{k} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$

We compute the magnitude

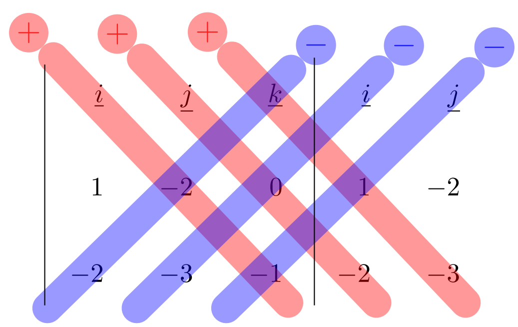
$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})} = \sqrt{2^2 + 1^2 + (-7)^2} = \sqrt{54} = 3\sqrt{6}$$

Hence, the area of the triangle PQR is $\frac{3\sqrt{6}}{2}$.

A useful way to visualise how to compute the cross product without remembering the nasty formula is as follows:



We multiply across diagonals: for the red diagonals we add the terms, for the blue diagonals we subtract terms. For example, for the cross product computation just performed to compute the area of the triangle we have:



$$= (-2)(-1)\underline{i} + 0.(-2)\underline{j} + 1.(-3)\underline{k} - (-2)(-2)\underline{k} - 0.(-3)\underline{i} - 1.(-1)\underline{j}$$

$$= 2\underline{i} + \underline{j} - 7\underline{k} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$