Feb 16. HW Solutions

Problem A

Since $\underline{u}, \underline{v}$ unit vectors then

$$\underline{u} = \cos\theta_1 \, \underline{i} + \sin\theta_1 \, \underline{j}$$

$$\underline{v} = \cos\theta_2 \, \underline{i} + \sin\theta_2 \, \underline{j}$$

Then, $$\underline{u} \cdot \underline{v} = (\cos\theta_1 \, \underline{i} + \sin\theta_1 \, \underline{j}) \cdot (\cos\theta_2 \, \underline{i} + \sin\theta_2 \, \underline{j})$$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

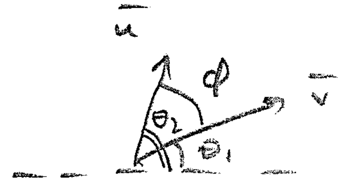
by dot product formula.

Also, using (geometric) cosine formula to compute dot product, we have

$$\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\varphi$$

where φ is angle between $\underline{u}$ and $\underline{v}$

Spse $\theta_1 \geq \theta_2$. Then,



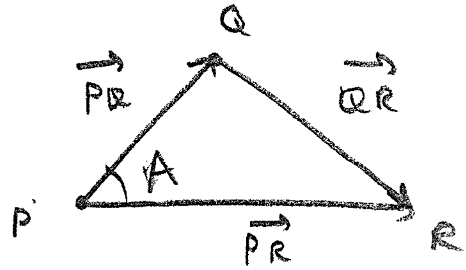
i.e. $\varphi = \theta_2 - \theta_1$

$$\Rightarrow \quad \underline{u} \cdot \underline{v} = \cos(\theta_1 - \theta_2)$$

Similarly if $\theta_2 > \theta_1$ then $\underline{u} \cdot \underline{v} = \cos(\theta_2 - \theta_1)$

$$= \cos(\theta_1 - \theta_2)$$

Problem B:



$|\overrightarrow{PQ}| = c$

$|\overrightarrow{PR}| = b$

$|\overrightarrow{QR}| = a$.

- $\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$

Hence;

$$\begin{aligned}
a^2 &= |\overrightarrow{QR}|^2 \\
&= \overrightarrow{QR} \cdot \overrightarrow{QR} \\
&= (\overrightarrow{PR} - \overrightarrow{PQ}) \cdot (\overrightarrow{PR} - \overrightarrow{PQ}) \\
&= |PR|^2 + |PQ|^2 - 2 \cdot \overrightarrow{PQ} \cdot \overrightarrow{PR} \\
&= b^2 + c^2 - 2 |\overrightarrow{PQ}| |\overrightarrow{PR}| \cos A \\
&= b^2 + c^2 - 2bc\cos A.
\end{aligned}$$

Feb 19: HW Solutions

Problem A:

A, B, C, D lie in same plane

$\iff$ displacement vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ all lie in common plane

$\iff$ parallelepiped spanned by $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ has volume 0

Check: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $\overrightarrow{AC} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, $\overrightarrow{AD} = \begin{bmatrix} 0 \\ -5 \\ 8 \end{bmatrix}$

$$\text{Volume} = \left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right|$$

$$= \left| \begin{bmatrix} 0 \\ 20 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -5 \\ 8 \end{bmatrix} \right| = 20 \neq 0$$

Hence, points don't lie in plane.

Problem B:

$$(\underline{u}+\underline{v}) \times (\underline{u}-\underline{v}) = \overset{0}{\cancel{\underline{u}\times\underline{u}}} + \underline{v}\times\underline{u} - \underline{u}\times\underline{v} - \overset{0}{\cancel{\underline{v}\times\underline{v}}}$$

$$= \underline{v} \times \underline{u}$$

Hence; since $\underline{v} \times \underline{u}$ $\perp$ to both $\underline{u}$ and $\underline{v}$ result follows.

Feb 26: HW solution

Problem A:

1) Multiply by r;

$$r^2 = 4r\cos\theta + 2r\sin\theta$$

$$\Rightarrow x^2+y^2 = 4x + 2y$$

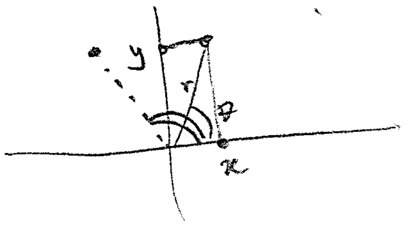
$$\Rightarrow x^2 - 4x + y^2 - 2y = 0$$

$$\Rightarrow (x-2)^2 + (y-1)^2 = 5$$

ie circle centred at $(2,1)$, radius $\sqrt{5}$

Problem B $\quad$ If $(x,y) = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j}$

where $r = \sqrt{x^2+y^2}$ then



• $(0 \le \theta \le \pi)$ $\quad (-x,y) = r\cos(\pi-\theta)\,\underline{i} + r\sin(\pi-\theta)\,\underline{j}$

$\underline{\text{ie}}$ $\quad (x,y) \longleftrightarrow (-x,y)$

corresponds to

$$(r,\theta) \longleftrightarrow (r, \pi-\theta)$$

in polar coords (assuming $0 \le \theta \le \pi$)

The equation

$$r = 1 + \sin\theta$$

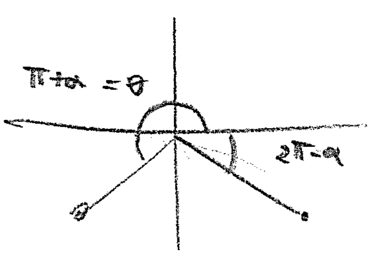
is invariant under $\theta \longleftrightarrow \pi - \theta$

$\underline{\text{ie}}$ $\quad r = 1 + \sin\theta$

$\iff r = 1 + \sin(\pi-\theta)$

using $\sin(\pi-\theta) = \sin\theta$, when $0 \le \theta \le \pi$.

• $(\pi \le \theta \le 2\pi)$ $\quad$ Similar, reflect in y-axis corresponds to transformation



$\theta \longleftrightarrow 3\pi - \theta$ $\quad$ and fact

that $\sin(3\pi-\theta) = \sin\theta$,

Feb 28. HW Solution

Problem A:

In Cartesian:

$$x = 0$$
$$y^2 + z^2 = 1$$

Spherical

$\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

$\rho = 1$ $\qquad \rho = 1$

$0 \leq \varphi \leq \pi$ $\qquad 0 \leq \varphi \leq \pi$

Cylindrical

$\theta = \frac{\pi}{2}$ or $\theta = 3\pi/2$

$r^2 + z^2 = 1$ $\qquad r^2 + z^2 = 1$

Problem B

Cartesian:

$$z \leq 0$$
$$x^2 + y^2 + z^2 \leq 4.$$

Spherical

$0 \leq \rho \leq 2$

$0 \leq \theta \leq 2\pi$

$\frac{\pi}{2} \leq \varphi \leq \pi$

Cylindrical

$0 \leq \theta \leq 2\pi$

$z \leq 0$

$r^2 + z^2 \leq 4$

Mar 2 : HW Solutions

A)

1) Turning points : $0 = \frac{d}{dt}(t^4 - 2t^3 + 1)$

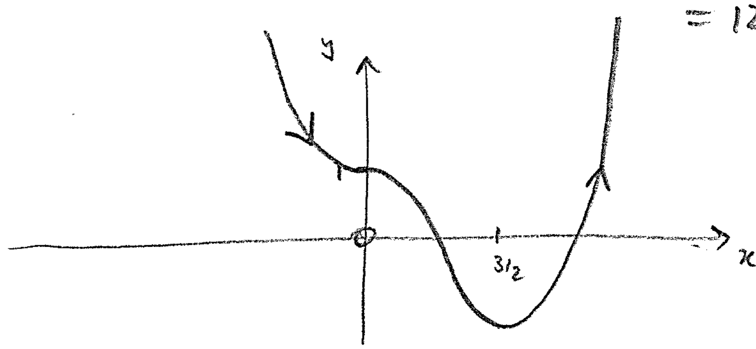
$$= 4t^3 - 6t^2$$

$$= 2t^2(2t-3)$$

i.e. $t=0$ and $t = 3/2$.

Second derivative test:

$$\frac{d}{dt}(4t^3 - 6t^2) = 12t^2 - 12t$$
$$= 12t(t-1) = \begin{cases} 0 & t=0 \\ >0 & t = 3/2. \end{cases}$$



2) $\underline{x}'(t) = \begin{bmatrix} 1 \\ 4t^3 - 6t^2 \end{bmatrix}$, $\underline{x}'(3) = \begin{bmatrix} 1 \\ 54 \end{bmatrix}$

3) $f(x) = x^4 - 6x^3 + 1$

4) $\underline{x}(t) = \begin{bmatrix} t \\ g(t) \end{bmatrix}$.

Problem B

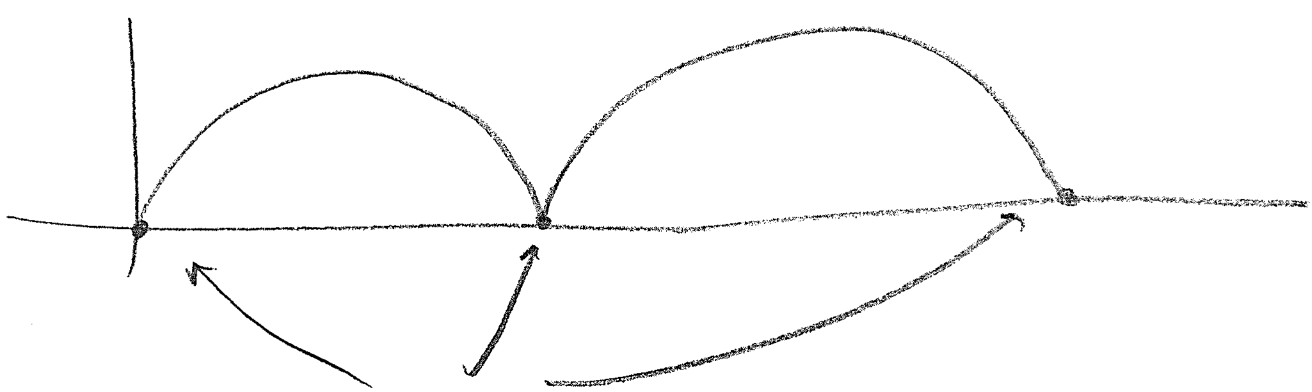
r = radius

1.) $$\underline{x}(t) = \begin{bmatrix} r(t - \sin(t)) \\ r(1 - \cos(t)) \end{bmatrix}$$

$$\underline{v}(t) = \underline{x}'(t) = \begin{bmatrix} r - r\cos(t) \\ r\sin(t) \end{bmatrix}$$

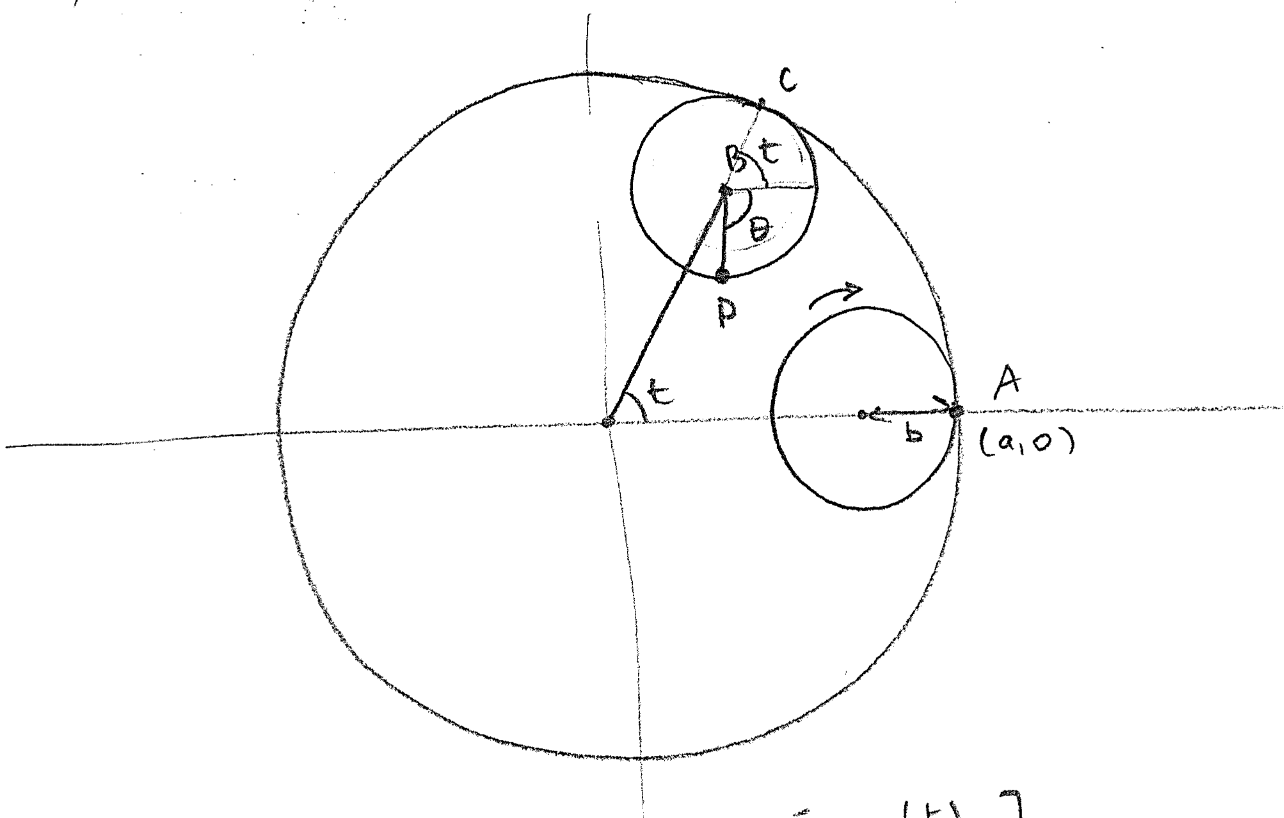
$$\underline{v}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{array}{ll} r(1-\cos(t)) = 0 \Leftarrow r = 2\pi k, & k \text{ integer} \\ r\sin(t) = 0 \Leftarrow r = k\pi, & k \text{ integer} \end{array}$$

Hence, $\underline{v}(t) = \underline{0} \iff t = 2\pi k$, k integer.



the 'turning points'.

2) $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$ $\qquad a = 6 \quad b = 5.$



$$\overrightarrow{OB} = \begin{bmatrix} (a-b)\cos(t) \\ (a-b)\sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$\overrightarrow{BP} = \begin{bmatrix} b\cos\theta \\ -b\sin\theta \end{bmatrix}$$

Need to determine θ in terms of t:

length of arc CP

= length of arc CA

$$\Rightarrow \quad b(t+\theta) = at$$

$$\Rightarrow \quad \theta = \left(\frac{a}{b} - 1\right)t = \frac{1}{5}t$$

$$\Rightarrow \quad \overrightarrow{BP} = \begin{bmatrix} 5\cos(\frac{1}{5}t) \\ -5\sin(\frac{1}{5}t) \end{bmatrix}$$

$$\Rightarrow \quad \overrightarrow{OP} = \begin{bmatrix} \cos(t) + 5\cos\left(\frac{1}{5}t\right) \\ \sin(t) - 5\sin\left(\frac{1}{5}t\right) \end{bmatrix}.$$