Middlebury College

Multivariable Calculus
Spring 2018
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February 14 Lecture

1 Dandelin's spheres

In this exercise we will determine the foci of an ellipse by a method known as *Dandelin's spheres* (discovered in 1822 by Germinal Pierre Dandelin, a Belgian mathematician).

Recall from February 12 Lecture the following results:

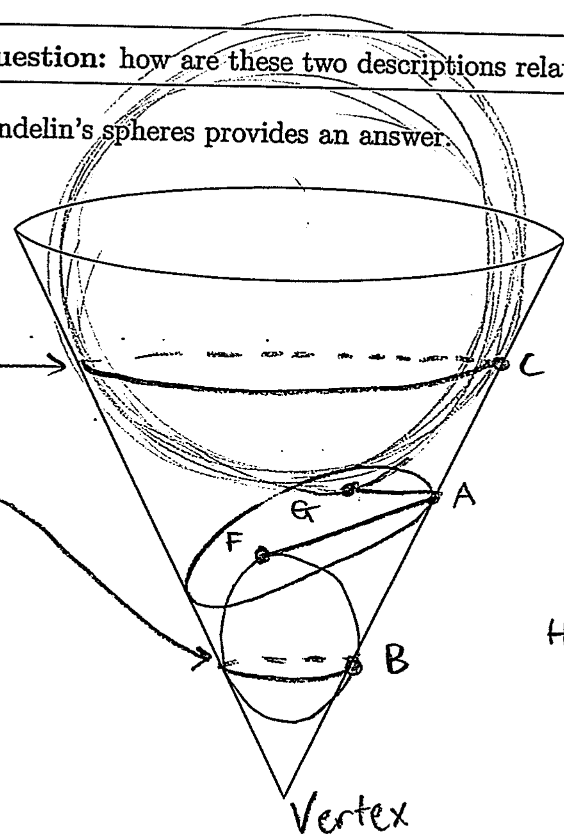
> 1. If we drop a ball S into a cone C then the set of points on the cone intersecting S is a circle at a fixed height from the vertex of C.
> 2. Let L and L' be two line segments emanating from a common point P that are both tangent to a ball S. Denote the single point of intersection of L (resp. L') with S by Q (resp. Q'). Then, L and L' have the same length.

An **ellipse** is a **conic section:** ellipses are obtained by intersecting a suitable plane with a cone. An ellipse is also a planar curve surrounding two fixed points (called **foci**) such that the sum of the distances to the two focal points is constant for every point on the ellipse.

> **Question:** how are these two descriptions related?

The method of Dandelin's spheres provides an answer.

By 1., these circles are at a fixed height.



• AB and AF are tangent to smaller ball $\Rightarrow$ by 2., $|AF| = |AB|$

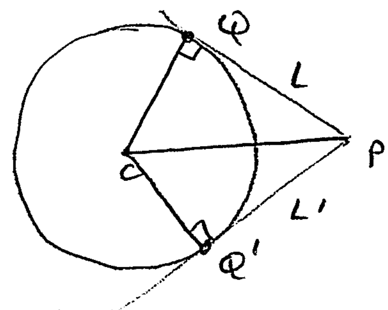
• Similarly, $|AC| = |AG|$

Hence,
$$|AF| + |AG| = |AB| + |AC| = |BC| \leftarrow \text{this is fixed quantity.}$$

Hence, Dandelin spheres provide geometric approach to determining foci

2 Displacement vectors In this paragraph we are going to show the (perhaps obvious?) result 2. indicated above - *Let L and L' be two line segments emanating from a common point P that are both tangent to a ball S. Denote the single point of intersection of L (resp. L') with S by Q (resp. Q'). Then, L and L' have the same length.*

Diagram:



We want to show that $|PQ| = |PQ'|$. It's sufficient to show that $|PQ|^2 = |PQ'|^2$.

1. Explain why we can assume that P is the origin in space.

In any geometric setting we may choose origin where we want (but will need to be consistent in a problem!).

2. Denote the center of the sphere by C. Explain why

$$|\overrightarrow{OC}|^2 = |\overrightarrow{OQ}|^2 + |\overrightarrow{QC}|^2 \quad \text{and} \quad |\overrightarrow{OC}|^2 = |\overrightarrow{OQ'}|^2 + |\overrightarrow{Q'C}|^2$$

L, L' are tangent to sphere $\Rightarrow CQ \perp L$ and $CQ' \perp L'$

Now use Pythagoras Thm.

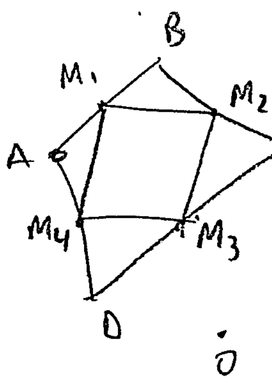
3. Use the previous result, and some algebra, deduce that $|\overrightarrow{OQ}|^2 = |\overrightarrow{OQ'}|^2$

Since $|QC|$ = radius of sphere = $|Q'C|$, we use previous result to get

$|OQ|^2 = |OQ'|^2$.

3 A mysterious paralleologram Let A, B, C, D be points in space such that they don't all lie on a common line. Let $Q = ABCD$ be the quadrilateral. Denote the mid points of the edges of Q by M_1, M_2, M_3, M_4.

Claim: The quadrilateral $M_1M_2M_3M_4$ is a paralleogram.



1. What do you need to show in order to prove the **Claim**?

Sufficient to show $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$ and $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$

2. Using displacement vectors, show that $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$

We have $\overrightarrow{OM_1} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{OM_2} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$

$$\Rightarrow \overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$$
$$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} - \overrightarrow{OA} - \frac{1}{2}\overrightarrow{AB}$$
$$= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}, \quad b/c \ \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$$
$$= \frac{1}{2}\overrightarrow{AC}$$

Similarly, can show $\overrightarrow{M_4M_3} = \frac{1}{2}\overrightarrow{AC}$.

3. Deduce the **Claim**.

Perform same calulation for $\overrightarrow{M_1M_4} = \overrightarrow{M_2M_3}$.