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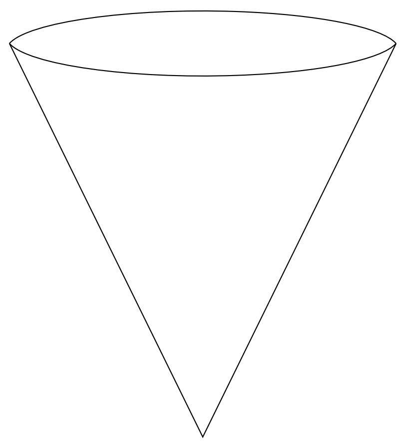
February 12 Lecture

In this exercise we will have a geometric stretch and begin to flex our mathematical muscles. For our workout we will investigate the determination of the foci of an ellipse by a method known as *Dandelin's spheres* (discovered in 1822 by Germinal Pierre Dandelin, a Belgian mathematician).

Warm-up exercises

1. [**Some geometry**] Consider the following geometric figure C, known as a **cone** (for obvious reasons!).



   (a) Imagine you dropped a ball S (mathematicians often call balls, '*spheres*') of radius r into the cone. Describe the **set** (= collection) of points $\mathcal{P}$ on the cone that will be touching the ball S.

   (b) Drop a larger ball S' of radius $s > r$ into the cone. Describe the set of points $\mathcal{P}'$ on the cone that will be touching the ball S'.

   (c) Can you state a geometric relationship between the sets $\mathcal{P}$ and $\mathcal{P}'$?

2. [**Some linear algebra**] Recall the **dot product**: let $\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$.

Then, the dot product is the real number

$$\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

(a) FILL IN THE BLANKS! Let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$ be vectors.

i.

$$\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{\hspace{5cm}}$$

ii.

$$(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{\hspace{5cm}}$$

iii. If $\underline{u} \cdot \underline{v} = 0$ then

$$(\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) = \underline{\hspace{5cm}}$$

What well-known Theorem is this?

(b) Suppose $n = 3$. Give a geometric interpretation of the quantity

$$|\underline{u}| \stackrel{def}{=} \sqrt{\underline{u} \cdot \underline{u}}$$

(c) Imagine a ball sitting in front of you. Choose a point P outside the ball. Imagine drawing a straight line L from P to a point Q on the ball so that the line L is tangent to the ball. Choose another point Q' on the ball ($Q \neq Q'$) given by drawing another line L' starting at P that is tangent to the ball. (*It may be useful to draw a picture below!*)

What is the relationship between the length of L and the length of L'?