MATH 223: PRACTICE EXAM III

1a) T – consequence of Clairaut

b) F – $f(x,y) = x$ has no extrema

c) F – $\int_0^1 \int_x^1 f(x,y)\,dy\,dx = \int_0^1 \int_0^y f(x,y)\,dx\,dy$

d) F –

e) T – definition

f) F – type error

g) ? –

h) T –

i) T

j) T – use linearisation of $f(x,y) = \sqrt{xy}$.

2a) $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$, $\overrightarrow{PR} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

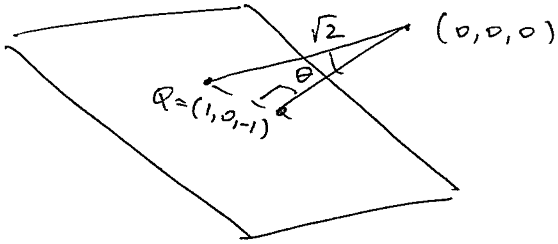
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$$

b) Π passes through $P = (2,1,1)$. Hence,

$$0 = -5(x-2) + 5(y-1) + 0.$$

$$\Rightarrow \quad -5x + 5y = -5$$

c)



Distance

$= \sqrt{2}\,|\cos\theta|$

$= \sqrt{2}\,\left|\left(\dfrac{\overrightarrow{OQ}\cdot \underline{n}}{|\overrightarrow{OQ}||\underline{n}|}\right)\right|$

where $\underline{n} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$ is normal

$\Rightarrow$ distance $= \cancel{\sqrt{2}}\left|\left(\dfrac{1\cdot(-5)+0\cdot 5+(-1)\cdot 0}{\cancel{\sqrt{2}}\cdot 5\sqrt{2}}\right)\right|$

$= \dfrac{1}{\sqrt{2}}$

3a) $\dfrac{\partial f}{\partial x} = x^2 - 2xy\,e^{-x^2} + 2y$

$\Rightarrow \quad f(x,y) = \int x^2 - 2xy\,e^{-x^2} + 2y \, dx$

$= \frac{1}{3}x^3 + y e^{-x^2} + 2yx + g(y)$

$e^{-x^2} + 2x + \cos(y) = \dfrac{\partial f}{\partial y} = e^{-x^2} + 2x + g'(y)$

$\Rightarrow \quad g'(y) = \cos(y)$

$\Rightarrow \quad g(y) = \sin y + C$

Hence, $\quad f(x,y) = \frac{1}{3}x^3 + ye^{-x^2} + 2xy + \sin(y) + C$

b) $\displaystyle\int_C \underline{F}\cdot dr = f(Q) - f(P)$, since $\underline{F} = \nabla f$

$= f(-1,0) - f(1,0)$

$= \frac{2}{3}$

4) a) Let $g(x,y,z) = x^2 + y^2 + z^2 - 6$

$h(x,y,z) = (x-3)^2 + y^2 + (z+1)^2 = 16$

$\nabla g(1,1,2) = \begin{bmatrix} 2 & 2 & 4 \end{bmatrix}$

$\nabla h(-1,0,-1) = \begin{bmatrix} -8 & 0 & 0 \end{bmatrix}$

Tangent plane to S_1:

$2(x-1) + 2(y-1) + 4(z-2) = 0$

$\Rightarrow 2x + 2y + 4z = 12$

$\Rightarrow x + y + 2z = 6$

Tangent plane to S_2:

$-8(x+1) = 0$

$\Rightarrow$ ~~[illegible]~~ . $x = -1$

b) Line is perpendicular to both plane normals

$\Rightarrow$ Line ~~perpendicular~~ parallel to

$$\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \times \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -32 \\ 16 \end{bmatrix}$$

parameterisation requires point on ~~plane~~ line:

$P = (a,b,c)$. $a + b + 2c = 6$

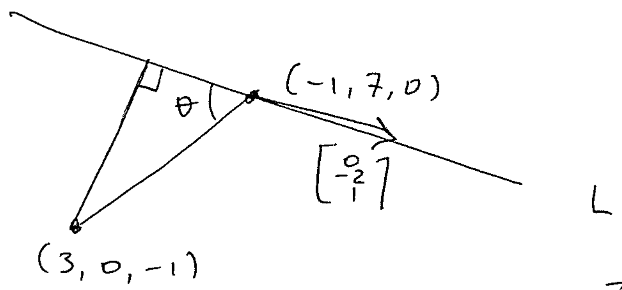
~~[illegible]~~
Must satisfy $a = -1$ and ~~[illegible]~~

$\Rightarrow b + 2c = 7$ eg $(-1, 7, 0)$ on line

Hence, parameterisation is

$$\begin{bmatrix} -1 \\ 7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

c)



Let $\underline{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 7 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}$

Then,

distance $= |\underline{u}|\ |\sin\theta|$

$= |\cancel{\underline{u}}| \cdot \left| \dfrac{\underline{u} \times \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}{|\cancel{\underline{u}}| \cdot \left|\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right|} \right|$

$= \dfrac{\left|\underline{u} \times \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right|}{\sqrt{5}}$

$= \dfrac{1}{\sqrt{5}} \left| \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \right|$

$= \dfrac{1}{\sqrt{5}} \left| \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right| = \dfrac{1}{\sqrt{5}} \sqrt{25 + 16 + 64}$

$= \dfrac{1}{\sqrt{5}} \sqrt{105}$

5) a) $\nabla f = \begin{bmatrix} 3x^2 - y & -2y - x \end{bmatrix}$

① $3x^2 - y = 0$

② $-2y - x = 0 \Rightarrow x = -2y$

① $\Rightarrow 3(-2y)^2 - y = 0$

$\Rightarrow 3.4y^2 - y = 0$

$\Rightarrow y(12y - 1) = 0 \Rightarrow y = 0$ or $y = 1/12$

If $\quad y=0 \quad$ then $\quad x=0$

$\quad\quad y=1/12 \quad$ then $\quad x=-1/6$

Hence, $(0,0)$ and $(-1/6, 1/12)$ are critical points

Compute

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 6x \\ \frac{\partial^2 f}{\partial x \partial y} &= -1 \\ \frac{\partial^2 f}{\partial y^2} &= -2 \end{aligned}\right\}$$

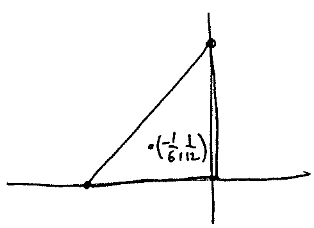
$(0,0)$: Hessian $\begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$

$\Rightarrow$ saddle

$(-1/6, 1/12)$: Hessian $\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$

$\rightarrow$ local max

b)



- $y=0$, $0 \le x \le 1$:

  $f(x,0) = x^3 + 1$ has max at $(0,0)$

- $x=0$, $0 \le y \le 1$:

  $f(0,y) = 1-y^2$ has max at $(0,0)$

- $y = 1+x$

$$f(x, 1+x) = x^3 - (1+x)^2 - x(1+x) + 1$$
$$= x^3 - 2x^2 - 3x$$

Then; $\frac{d}{dx} f(x, 1+x)$

$$= 3x^2 - 2(1+x) - (1+2x)$$

$$= 3x^2 - 2 - 2x - 1 - 2x$$

$$= 3x^2 - 4x - 3 = 0$$

$$x = \frac{2 \pm \sqrt{16 + 4.9}}{6} = \frac{1}{3} \pm \frac{\sqrt{52}}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

$\Rightarrow x = \frac{1}{3}(1-\sqrt{13}) \in [-1, 0]$.

At $x = \frac{1}{3}(1-\sqrt{13})$ $\quad y = \frac{2}{3} - \frac{\sqrt{13}}{3}$

Hence, max on boundary can occur at $(0,0)$,

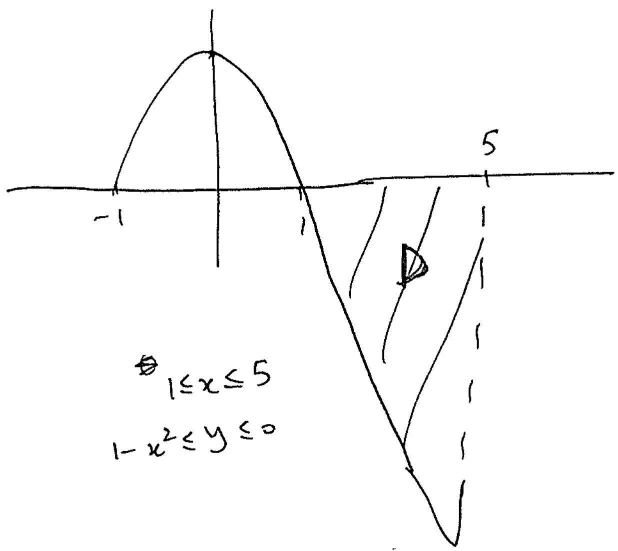
or $\left(\frac{1}{3}(1-\sqrt{13}), \frac{1}{3}(2-\sqrt{13})\right)$

$$f\left(\tfrac{1}{3}(1-\sqrt{13}), \tfrac{1}{3}(2-\sqrt{13})\right) = \frac{1}{3^3}(1-\sqrt{13})^3 - \frac{2}{3^2}(1-\sqrt{13})^2 - \frac{3}{3}(1-\sqrt{13})$$

$$< 1$$

Hence, max is at $(0,0)$.

6a)



$1 \le x \le 5$

$1 - x^2 \le y \le 0$

$$\iint_D f\,dA$$

$$= \int_{x=1}^{x=5} \int_{y=1-x^2}^{y=0} x + y \, dy\, dx$$

$$= \int_{x=1}^{x=5} \left[xy + \frac{1}{2}y^2 \right]_{1-x^2}^{0} dx$$

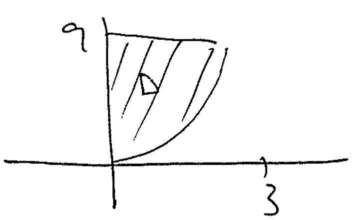
$$= \int_{x=1}^{x=5} \left[-x(1-x^2) - \frac{1}{2}(1-x^2)^2 \right] dx$$

$$= \int_{x=1}^{x=5} -\frac{1}{2} - x + x^2 + x^3 - \frac{1}{2}x^4 \, dx$$

$$= \left[-\frac{1}{2}x - \frac{1}{2}x^2 + \frac{x^3}{3} + \frac{x^4}{4} - \frac{1}{10}x^5 \right]_1^5$$

$$= \left(-\frac{11}{10} + \frac{7}{12} - \frac{5}{2} - \frac{25}{2} + \frac{125}{3} + \frac{625}{4} - \frac{3125}{10} \right)$$

b)



$$\int_{x=0}^{x=3} \int_{y=x^2}^{y=9} x e^{-y^2} \, dy\, dx = \int_{y=0}^{y=9} \int_{x=0}^{x=\sqrt{y}} x e^{-y^2} \, dx\, dy$$

$$= \int_{y=0}^{y=9} \left[\tfrac{1}{2} x^2 e^{-y^2} \right]_0^{\sqrt{y}} dy$$

$$= \int_{y=0}^{y=9} \tfrac{1}{2} y e^{-y^2} dy = \left[-\tfrac{1}{4} e^{-y^2} \right]_0^9$$

$$= -\tfrac{1}{4} \left(e^{-81} - 1 \right)$$

7a) points where C is parallel to level curves; there are two of them

b) We want to maximize $f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= x + y$$

subject to constraint

$$x^2 + 4y^2 = 1.$$

Use method of Lagrange multipliers:

Let $f(x,y) = x + y$

$g(x,y) = x^2 + 4y^2 - 1$

Require $\nabla f = \lambda \nabla g$, for some (x, y, λ):

ie ① $1 = \lambda 2x$ $\quad \lambda = \frac{1}{2x}$

② $1 = 8 \lambda y$ $\quad \Rightarrow \quad 1 = 8 \frac{y}{2x} \Rightarrow x = 4y$

③ $1 = x^2 + 4y^2$ $\quad$ ③ $\Rightarrow 1 = x^2 + 4y^2$

$$= (4y)^2 + 4y^2$$

$$= 20y^2$$

$$\Rightarrow y = \pm \frac{1}{2\sqrt{5}}$$

Hence, $\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}} \right)$ and $\left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}} \right)$

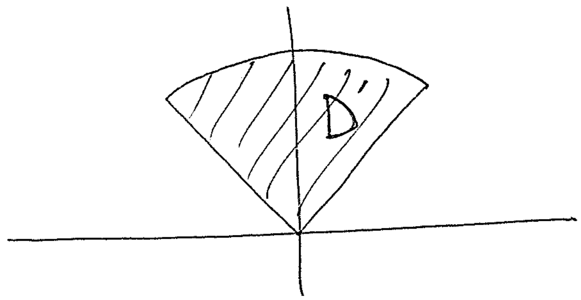
are potential constrained extrema.

We check: $f\left(\frac{2}{\sqrt5}, \frac{1}{2\sqrt5}\right) = \frac{1}{\sqrt5}\left(\frac{5}{2}\right)$

$$> f\left(-\frac{2}{\sqrt5}, -\frac{1}{2\sqrt5}\right)$$

Hence, $P = (x_0, y_0) = \left(\frac{2}{\sqrt5}, \frac{1}{2\sqrt5}\right) \in E$ maximises $\overrightarrow{OP} \cdot \underline{v}$.

8a)



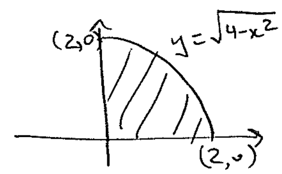
"rotate by $\frac{\pi}{4}$ counter-clockwise"

b)

Define $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\underline{x} \longmapsto \underbrace{\begin{bmatrix} 1/\sqrt2 & -1/\sqrt2 \\ 1/\sqrt2 & 1/\sqrt2 \end{bmatrix}}_{M} \underline{x} = \begin{bmatrix} 1/\sqrt2\,(x-y) \\ 1/\sqrt2\,(x+y) \end{bmatrix}$$

Then; if D:



then $L(D) = D'$

Also:

$(f \circ L)(x,y) = f\left(\frac{1}{\sqrt2}(x-y), \frac{1}{\sqrt2}(x+y)\right)$

$= \sqrt2\, y$

Hence; by change of coordinates

$$\iint_{D'} f\, dA = \iint_{D} (f \circ L)\,|\det M|\, dA$$

$$= \iint_{D} \sqrt2\, y\, dA$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} \sqrt2\, y\, dy\, dx = \int_{x=0}^{x=2} \left[\frac{\sqrt2}{2} y^2\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \frac{1}{\sqrt{2}} (4 - x^2)\, dx = \frac{1}{\sqrt{2}} \left[4x - \frac{x^3}{3} \right]_0^2$$

$$= \frac{1}{\sqrt{2}} \left[8 - \frac{8}{3} \right] = \frac{8\sqrt{2}}{3}$$