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# Practice Examination III

**Instructions:**

- You *must* attempt Problem 1.
- Please attempt at least five of Problems 2, 3, 4, 5, 6, 7, 8.
- If you attempt all eight problems then your final score will be the sum of your score for Problem 1 and the scores for the five remaining problems receiving the highest number points.
- Calculators are not permitted.

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1. (20 points) True/False:

   (a) Let $\underline{F}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$. If $\underline{F}$ is conservative then $u_y = v_x$.

   (b) Every function $f(x,y)$ defined on $\mathbb{R}^2$ has a local maximum or local minimum.

   (c) If $f(x,y)$ is a continuous function then $\int_0^1 \int_x^1 f(x,y)dydx = \int_0^1 \int_y^1 f(x,y)dxdy$

   (d) If $C$ is a closed oriented curve and $\underline{F}$ is a vector field satisfying $\int_C \underline{F} \cdot d\underline{r} = 0$ then $\underline{F}$ is conservative.

   (e) If $\underline{u}, \underline{v}$ are vectors then $|\underline{u} \times \underline{v}|$ is the area of the parallelogram spanned by $\underline{u}, \underline{v}$.

   (f) $3\nabla f = \frac{d}{dt} f(x+t, y+t, z+t)$

   (g) The vector line integral of $\underline{F}$ along the ellipse $x^2 + 5y^2 = 1$ is zero.

   (h) If $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^3$ lie in a common plane then $\underline{u} \cdot (\underline{v} + \underline{u}) \times \underline{w} = 0$.

   (i) Consider the surface $S: \ z^2 = f(x,y)$. If $P = (x, y, \sqrt{f(x,y)})$ is a point on $S$ with maximal distance from $(0,0,0)$ then $P$ is a local maximum of $g(x,y) = x^2 + y^2 + f(x,y)$.

   (j) Using linear approximation, the value $\sqrt{101 \cdot 10002}$ is estimated as $1000 + 5 + \frac{1}{10}$.

2. Let $P = (2,1,1)$, $Q = (1,0,-1)$, $R = (0,-1,2)$.

   (a) Compute $\overrightarrow{PQ} \times \overrightarrow{PR}$.

   (b) Write down the equation of the plane $\Pi: \ ax + by + cz = d$ containing the points $P, Q, R$.

   (c) Find the distance from the origin to $\Pi$.

3. Let $\underline{F}(x,y) = \begin{bmatrix} x^2 - 2xye^{-x^2} + 2y \\ e^{-x^2} + 2x + \cos(y) \end{bmatrix}$.

   (a) Show that $\underline{F}$ is conservative by finding a potential function $f(x,y)$ such that $\nabla f = \underline{F}$.

   (b) If $C$ is the oriented curve going from $(1,0)$ to $(-1,0)$ along the semicircle $x^2 + y^2 = 1$, $y \geq 0$, evaluate $\int_C \underline{F} \cdot d\underline{r}$

4. Consider the spheres

$$S_1: \ x^2 + y^2 + z^2 = 6, \qquad S_2: \ (x-3)^2 + y^2 + (z+1)^2 = 16.$$

(a) Determine the tangent plane to $S_1$ at the point $(1,1,2)$ and the tangent plane to $S_2$ at the point $(-1,0,-1)$

(b) Find a parameterisation of the line of intersection $L$ of the tangent planes.

(c) Determine the distance from the centre of $S_2$ to $L$.

5. (a) Classify the critical points of the function

$$f(x,y) = x^3 - y^2 - xy + 1$$

(b) Determine the absolute maximum of $f(x,y)$ on the triangle having vertices $(0,0)$, $(-1,0)$, $(0,1)$ *Hint: consider the extrema of $f(x,y)$ on the interior of the triangle and on the boundary of the triangle.)*

6. (a) Let $f(x,y) = x + y$ and $D$ be the region bounded between the $x$-axis and and the parabola $y = 1 - x^2$, $1 \le x \le 5$. Compute

$$\int \int_D f dA$$

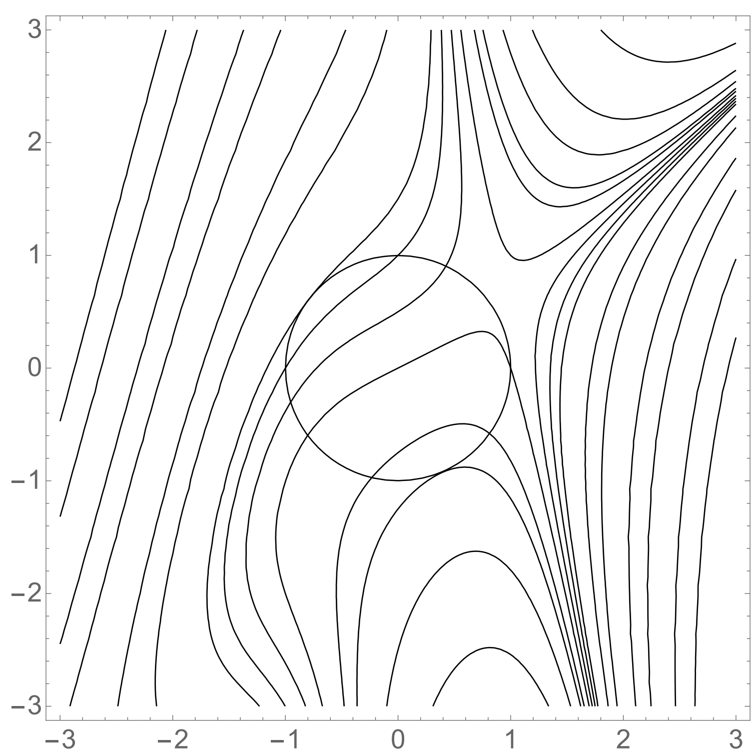
(b) Evaluate the integral by changing the order of integration

$$\int_0^3 \int_{x^2}^9 x e^{-y^2} dy dx$$

7. (a) Given below is the level curve diagram of a function $f(x,y)$. Mark the points on the circle $C$ where the extrema to the constrained optimisation problem

$$\begin{array}{l}\text{max. } f(x,y)\\ \text{subject to } C\end{array}$$

can occur.




(b) Let $\underline{v} = \begin{bmatrix}1\\1\end{bmatrix}$. Find the point $P = (x_0, y_0)$ on the ellipse $E: \; x^2 + 4y^2 = 1$ such that $\overrightarrow{OP} \cdot \underline{v}$ is maximised.

8. (a) Draw the region $D'$, described in polar coordinates by $\pi/4 \leq \theta \leq 3\pi/4$, $0 \leq r \leq 2$.

(b) Let $f(x,y) = y - x$. Using the linear change of coordinate formula, compute

$$\int\int_{D'} f dA$$

*(Hint: if* $\theta \in [0, 2\pi]$ *then* $M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *is the matrix corresponding to the 'rotate by* $\theta$ *counterclockwise' linear transformation.)*