Middlebury College

Math 223AB: Spring 2018
EXAMINATION II

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Sign the Honor Code Pledge below.
- Write your name on this exam and any extra sheets you hand in.
- You will have 120 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least three** of Problems 2, 3, 4, 5.
- Your final score will be the sum of your score for Problem 1 and the highest possible score obtained from three of the four remaining problems.
- There are **4 blank pages** attached for scratchwork and/or additional space for solutions.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	10 /10
QUESTION 2:	20 /20
QUESTION 3:	20 /20
QUESTION 4:	20 /20
QUESTION 5:	20 /20
TOTAL:	70 /70

NAME: C. F. GAUSS

"I have neither given nor received unauthorized aid on this assignment"

1. (10 points) True/False (no justification required)

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{3x^2+y^2} = 0$$

(b) If $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 1$ then the graph of $f(x, y)$ is a line.

(c) For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and any unit vector $\underline{u} \in \mathbb{R}^2$, $D_{\underline{u}}f(a, b) = -D_{-\underline{u}}f(a, b)$.

(d) The vector field $\underline{F} = \begin{bmatrix} x^2 + \frac{y}{x} \\ \frac{x}{y} + y^2 \end{bmatrix}$ admits a potential function.

(e) If $f(x, g(x)) = 0$ then $g'(x) = -f_x/f_y$ provided $f_y \neq 0$.

Solution: Write T(rue) or F(alse) in the corresponding box below

a) T	b) F	c) T	d) F	e) T

a) Use polar coords:

$$\frac{x^2y^2}{3x^2+y^2} = \frac{r^2\cos^2\theta\sin^2\theta}{3\cos^2\theta+\sin^2\theta} \to 0 \text{ as } r \to 0$$

b) Graph is a surface

c) $D_{\underline{u}} f(a,b) = \nabla f(a,b)\, \underline{u}$

d) $\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$

e) Let $w = f(x, g'(x))$.

$$0 = \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}\cdot 1 + \frac{\partial f}{\partial y}\cdot\frac{dg}{dx}$$
$$= f_x + f_y\, g'(x)$$

2. Let $f(u,v)$ a differentiable function, and

$$g : \{(x,y) \mid y \neq 0\} \to \mathbb{R}^2 \ , \ g(x,y) = \begin{bmatrix} x/y \\ xy \end{bmatrix}$$

(a) (5 points) Determine $Dg(x,y)$.

(b) (5 points) Let $h = f \circ g$. Determine $\frac{\partial h}{\partial x}(1,1)$ and $\frac{\partial h}{\partial y}(1,1)$ as a function of $f_u(1,1)$ and $f_v(1,1)$.

(c) (10 points) Let $q(x,y,z)$ be a differentiable scalar-valued function and define $p(u,v,w) = q(u-v, v-u, w-u)$. Show that

$$\frac{\partial p}{\partial u} + \frac{\partial p}{\partial v} + \frac{\partial p}{\partial w} = 0$$

a) $$Dg(x,y) = \begin{bmatrix} 1/y & -x/y^2 \\ y & x \end{bmatrix}$$

b) $$\frac{\partial h}{\partial x}(1,1) = \frac{\partial}{\partial x}(f \circ g)(1,1) \quad . \quad \text{Use Chain Rule}$$

$$D(f \circ g)(1,1) = Df(g(1,1))\, Dg(1,1)$$

$$= \nabla f(1,1) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \Big[\underbrace{f_u(1,1) + f_v(1,1)}_{\frac{\partial h}{\partial x}(1,1)} \quad \underbrace{-f_u(1,1) + f_v(1,1)}_{\frac{\partial h}{\partial y}(1,1)}\Big]$$

$\rightarrow$

$$\frac{\partial h}{\partial x}(1,1) = f_u(1,1) + f_v(1,1)$$

$$\frac{\partial h}{\partial y}(1,1) = -f_u(1,1) + f_v(1,1)$$

$x = u - v$

$y = v - w$

$z = w - u$

c) $$\frac{\partial p}{\partial u} = \frac{\partial q}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial q}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial q}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$= q_x - q_z$$

$$\frac{\partial p}{\partial v} = \frac{\partial q}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial q}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial q}{\partial z} \cdot \frac{\partial z}{\partial v}$$

$$= -q_x + q_y$$

$$\frac{\partial p}{\partial w} = \frac{\partial q}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial q}{\partial y} \cdot \frac{\partial y}{\partial w} + \frac{\partial q}{\partial z} \cdot \frac{\partial z}{\partial w}$$

$$= -q_y + q_z$$

$$\frac{\partial p}{\partial u} + \frac{\partial p}{\partial v} + \frac{\partial p}{\partial w} = (q_x - q_z) + (-q_x + q_y) + (-q_y + q_z) = 0 .$$

3. Consider the surface

$$S: \quad x^4 - y^4 + z^4 = 16$$

(a) (10 points) Determine the tangent plane to S at $(1,1,2)$.

(b) (10 points) Consider the C curve obtained as the intersection of S with the plane $x+y+z=4$. Determine the tangent line to C at $(1,1,2)$.

a) Let $h(x,y,z) = x^4 - y^4 + z^4$

Then, normal for tangent plane is parallel to $\nabla h(1,1,2)^t = \begin{bmatrix} 4 \\ -4 \\ 32 \end{bmatrix}$ ie $\begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$

$\Rightarrow$ tangent plane is

$$(x-1) - (y - 1) + 8(z-2) = 0$$

$\Rightarrow$ ~~x+y~~ $x - y + 8z = 16$

b) C lies in S and C lies in $x+y+z=4$

$\Rightarrow$ tangent line is perpendicular to

$$\begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ tangent line parallel to $\begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -9 \\ 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -9 \\ 7 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$$

4. Consider the conservative vector field

$$\underline{F} = \begin{bmatrix} e^x y \\ e^x + 2y \end{bmatrix}$$

(a) (10 points) Determine the potential function $f(x, y)$ for $\underline{F}$ satisfying $f(0,0) = 1$.

(b) (10 points) Determine the tangent line to the level curve $f(x, y) = 1$ at $(0,0)$.

a) $$e^x y = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = \int e^x y \, dx$$
$$= e^x y + g(y)$$

$$e^x + 2y = \frac{\partial f}{\partial y} = e^x + g'(y)$$
$$\Rightarrow \quad g'(y) = 2y \quad \Rightarrow g(y) = \int 2y \, dy$$
$$= y^2 + C$$

~~Require~~ Hence, $f(x,y) = e^x y + y^2 + C$

Require $1 = f(0,0) = C$

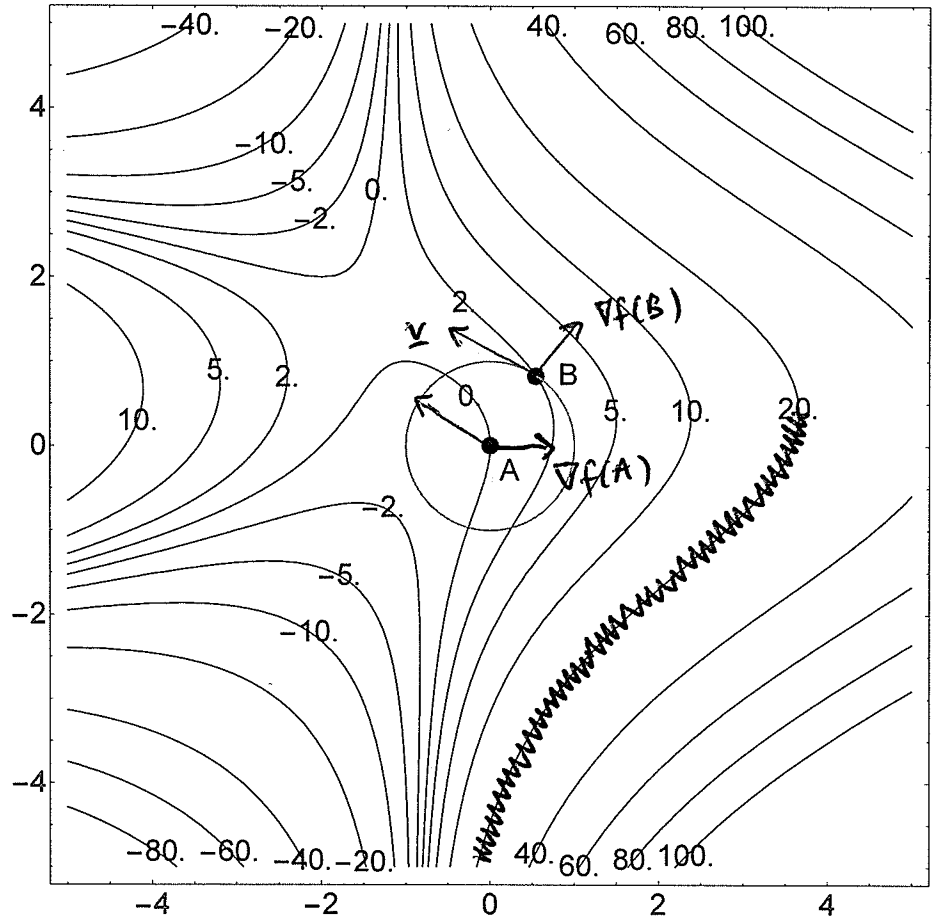
$$\Rightarrow \quad f(x,y) = e^x y + y^2 + 1$$

b) $$\nabla f(0,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Tangent line is perpendicular to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and passes through $(0,0)$

$\Rightarrow$ $\boxed{y = 0}$ is tangent line to $f = 1$ at $(0,0)$.

5. The level curve diagram of a function $f(x, y)$ is given below. Plotted are the points $A = (0, 0)$ and $B = (1/2, \sqrt{3}/2)$. It is known that $|\nabla f(A)| = |\nabla f(B)|$.



(a) (5 points) On the level curve diagram mark the portion(s) of the level curve $f = 20$ where $f_x \geq 0$ and $f_y \leq 0$.

(b) (5 points) Give an example of a unit vector $\underline{u} \in \mathbb{R}^2$ such that $D_{\underline{u}} f(A) \geq 0$ and $D_{\underline{u}} f(B) < 0$.

Solution:

$$\underline{u} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(c) (10 points) The unit circle centred at A is plotted. Suppose that $\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$, $t \in [0, 2\pi]$, is a parameterisation of the circle so that $B = \underline{x}(\pi/3)$. Let $\underline{v} = \underline{x}'(\pi/3)$.

Based on the information given in the level curve diagram, mark the correct relationship between $D_{\underline{v}} f(A)$ and $D_{\underline{v}} f(B)$:

$D_{\underline{v}} f(A) < D_{\underline{v}} f(B)$	$D_{\underline{v}} f(A) = D_{\underline{v}} f(B)$	$D_{\underline{v}} f(A) > D_{\underline{v}} f(B)$
X		

Since angle between $\underline{v}$ and $\nabla f(A)$ is larger than angle between $\underline{v}$ and $\nabla f(B)$.