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Practice Examination II Solution

Instructions:

- You *must* attempt Problem 1.
- Please attempt at least three of Problems 2,3,4,5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the scores for the three remaining problems receiving the highest number points.
- Calculators are not permitted.

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1. (10 points) True/False:

   (a)
   $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^3} = 1$$

   (b) If the partial derivatives of $f(\underline{x})$ exist at $\underline{a}$ then f is differentiable at $\underline{a}$.

   (c) If $\frac{\partial f}{\partial x}(P) = 0$ then $f(\underline{x}) \le f(P)$ for all $\underline{x}$ in a small disc centred at P.

   (d) Let $f(x,y)$ be function such that $f(tx,ty) = t^3 f(x,y)$, for all t. Then,
   $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y)$$

   (e) The vector field $\underline{F} = \begin{bmatrix} x^2\sin(xy) \\ y^2\cos(xy)\end{bmatrix}$ is conservative.

   Solution:

   (a) F

   (b) F

   (c) F

   (d) T

   (e) F

2. Consider the conservative vector field on $\mathbb{R}^2$
   $$\underline{F} = \begin{bmatrix} 3x^2y + y^3 + 1 \\ x^3 + 3xy^2 + 2\end{bmatrix}$$

   (a) Determine the potential function $f(x,y)$ for $\underline{F}$ satisfying $f(-1,0) = 0$.

   (b) Determine the tangent line to the level curve $f(x,y) = 0$ at $(-1,0)$.

(c) Show that the tangent line to the level curve $f(x,y) = 3$ at $(0,1)$ is parallel to the tangent line computed in (b).

Solution:

(a)

$$\frac{\partial f}{\partial x} = 3x^2y + y^3 + 1 \implies f(x,y) = \int 3x^2y + y^3 + 1dx = x^3y + xy^3 + x + g(y)$$

$$x^3 + 3xy^2 + 2 = \frac{\partial f}{\partial y} = x^3 + 3xy^2 + g'(y) \implies g'(y) = 2 \implies g(y) = 2y + C$$

Hence, potential function is of the form

$$f(x,y) = x^3y + xy^3 + x + 2y + C.$$

The condition

$$0 = f(-1,0) = -1 + C \implies C = 1$$

Hence,

$$f(x,y) = x^3y + xy^3 + x + 2y + 1$$

(b) We have

$$\nabla f = \underline{F} = \begin{bmatrix} 3x^2y + y^3 + 1 \\ x^3 + 3xy^2 + 2 \end{bmatrix} \implies \nabla f(-1,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then tangent is perpendicular to $\nabla f(-1,0)$ so it has slope -1. Hence,

$$y = (-1)(x+1) = -x - 1$$

is the equation of the tangent line.

(c) Since

$$\nabla f(0,1) = \underline{F}(0,1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and $\nabla f(0,1)$ is parallel to $\nabla f(-1,0)$, we must have the tangent line to $f = 3$ is parallel to the tangent line in (b).

3. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x,y) \mapsto x^2y^2 - x - y$$

(a) Determine $\nabla f(3,2)$

(b) Determine the equation of the tangent plane to the graph of $f(x,y)$ at $(3,2,31)$.

(c) Use a linear approximation to find the approximate value of $f(2.9, 2.1)$.

(d) Compute the directional derivative of f at $(3,2)$ in the direction $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solution:

(a) We have

$$\nabla f = \begin{bmatrix} 2xy^2 - 1 & 2x^2y - 1 \end{bmatrix}$$

Hence, $\nabla f(3,2) = \begin{bmatrix} 23 & 35 \end{bmatrix}$.

(b) Use linearisation

$$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) \quad \Longrightarrow \quad L(x,y) = 31 + 23(x-3) + 35(y-2)$$

Then, tangent plane to the graph $f(x,y)$ at $(3,2,31)$ is the graph of $L(x,y)$:

$$z = 31 + 23(x-3) + 35(y-2)$$

(c) Using linearisation we have

$$L(2.9, 2.1) = 31 + 23.(-0.1) + 35.(0.1) = 32.2$$

(d) Let $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Then,

$$D_{\underline{v}} f(3,2) = \nabla f(3,2)\underline{v} = \begin{bmatrix} 23 & 35 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 139$$

4. Consider the differentiable functions

$$f : \{(x,y) \mid x > 0\} \to \mathbb{R}^2 \, , \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y/x \\ x^2 + y^2 \end{bmatrix}$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \, , \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2xy \\ x^2 - y^2 \end{bmatrix}$$

$$h : \mathbb{R}^2 \to \mathbb{R} \, , \ (x,y) \mapsto x^3 + 3xy + y^3$$

(a) Compute the Jacobian matrices Df, Dg, Dh.

(b) Let $p(x,y) = g(y/x, x^2 + y^2)$. Determine Dp.

(c) Let $q(x,y) = h(2xy, x^2 - y^2)$. Determine ∇q and compute $\frac{\partial q}{\partial y}(1,1)$.

Solution:

(a)

$$Df = \begin{bmatrix} -y/x^2 & 1/x \\ 2x & 2y \end{bmatrix}, \quad Dg = \begin{bmatrix} 2y & 2x \\ 2x & -2y \end{bmatrix}, \quad Dh = \begin{bmatrix} 3x^2 + 3y & 3x + 3y^2 \end{bmatrix}$$

(b) We can write $p(x,y) = (g \circ f)(x,y)$. Hence,

$$Dp(x,y) = D(g \circ f)(x,y) = (Dg)(f(x,y))Df(x,y)$$

$$= \begin{bmatrix} 2(x^2+y^2) & 2y/x \\ 2y/x & -2(x^2+y^2) \end{bmatrix} \begin{bmatrix} -y/x^2 & 1/x \\ 2x & 2y \end{bmatrix} = \begin{bmatrix} -2y - 2y^3/x^2 & 2x + 6y^2 x \\ -2y^2 x^3 - 4x(x^2+y^2) & 2y/x^2 - 4y(x^2+y^2) \end{bmatrix}$$

(c) We can write $q(x,y) = (h \circ g)(x,y)$. Hence,

$$\nabla q = Dq = D(h \circ g) = (Dh)(g(x,y))Dg(x,y)$$

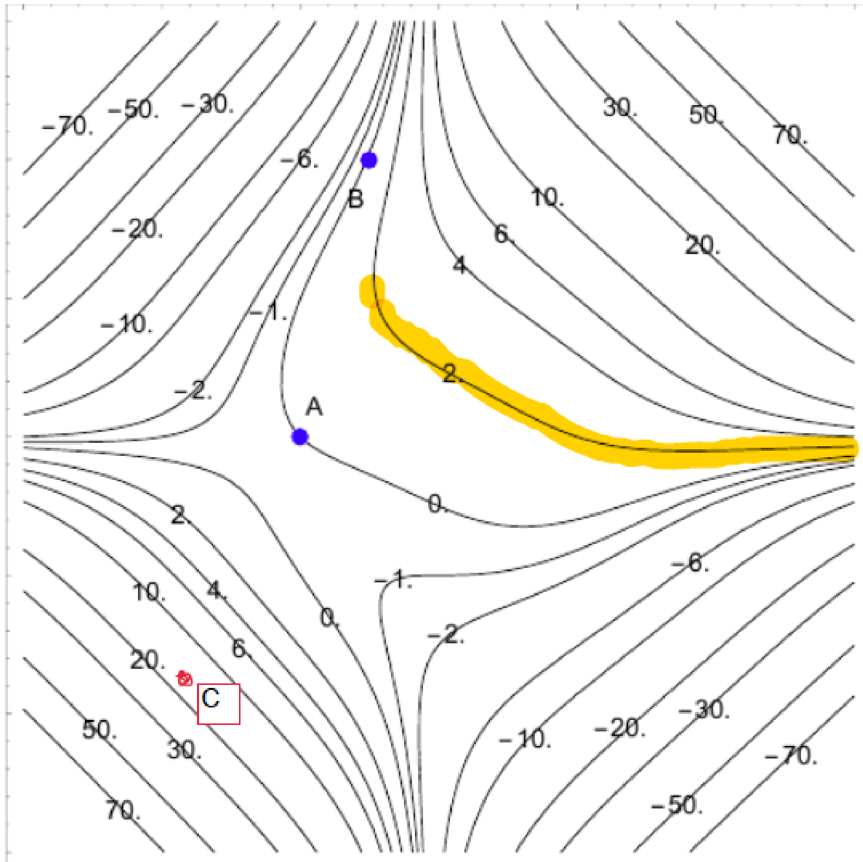
$$= \begin{bmatrix} 12x^2y^2 + 3(x^2 - y^2) & 6xy + 3(x^2 - y^2)^2 \end{bmatrix} \begin{bmatrix} 2y & 2x \\ 2x & -2y \end{bmatrix}$$

$$= \begin{bmatrix} 24x^2y^3 + 12x^2y - 6y^3 + 6x(x^2-y^2)^2 & 24x^3y^2 + 6x^3 - 18xy^2 - 6y(x^2-y^2)^2 \end{bmatrix}$$

Hence, since $\nabla q(1,1) = \begin{bmatrix} 30 & 12 \end{bmatrix}$,

$$\frac{\partial q}{\partial y}(1,1) = 12$$

5. A function $f(x, y)$ has the following level curve diagram



(a) On the level curve diagram mark the portion(s) of the level curve $f = 2$ where $\frac{\partial f}{\partial y} \geq 0$.

(b) Based on the information given in the level curve diagram, mark the correct relationship between $|\nabla f(A)|$ and $|\nabla f(B)|$:

$\lvert\nabla f(A)\rvert < \lvert\nabla f(B)\rvert$	$\lvert\nabla f(A)\rvert = \lvert\nabla f(B)\rvert$	$\lvert\nabla f(A)\rvert > \lvert\nabla f(B)\rvert$
X		

(c) Let $\underline{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. On the level curve diagram mark a point C satisfying the following conditions:

$$\frac{\partial f}{\partial x}(C) < 0, \quad \text{and} \quad D_{\underline{v}} f(C) > 0$$