Middlebury College

Math 223AB: Spring 2018

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Practice Examination II

Instructions:

- You *must* attempt Problem 1.
- Please attempt at least three of Problems 2,3,4,5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the scores for the three remaining problems receiving the highest number points.
- Calculators are not permitted.

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1. (10 points) True/False:

   (a)
   $$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^3} = 1$$

   (b) If the partial derivatives of $f(\underline{x})$ exist at $\underline{a}$ then f is differentiable at $\underline{a}$.

   (c) If $\frac{\partial f}{\partial x}(P) = 0$ then $f(\underline{x}) \leq f(P)$ for all $\underline{x}$ in a small disc centred at P.

   (d) Let $f(x,y)$ be function such that $f(tx,ty) = t^3 f(x,y)$, for all t. Then,
   $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f(x,y)$$

   (e) The vector field $\underline{F} = \begin{bmatrix} x^2\sin(xy) \\ y^2\cos(xy) \end{bmatrix}$ is conservative.

2. Consider the conservative vector field on $\mathbb{R}^2$
   $$\underline{F} = \begin{bmatrix} 3x^2y + y^3 + 1 \\ x^3 + 3xy^2 + 2 \end{bmatrix}$$

   (a) Determine the potential function $f(x,y)$ for $\underline{F}$ satisfying $f(-1,0) = 0$.

   (b) Determine the tangent line to the level curve $f(x,y) = 0$ at $(-1,0)$.

   (c) Show that the tangent line to the level curve $f(x,y) = 3$ at $(0,1)$ is parallel to the tangent line computed in (b).

3. Consider the function
   $$f : \mathbb{R}^2 \to \mathbb{R}\ ,\ (x,y) \mapsto x^2y^2 - x - y$$

   (a) Determine $\nabla f(3,2)$

   (b) Determine the equation of the tangent plane to the graph of $f(x,y)$ at $(3,2,31)$.

   (c) Use a linear approximation to find the approximate value of $f(2.9,2.1)$.

(d) Compute the directional derivative of f at $(3,2)$ in the direction $\underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

4. Consider the differentiable functions

$$f : \{(x,y) \mid x > 0\} \to \mathbb{R}^2 \,, \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y/x \\ x^2 + y^2 \end{bmatrix}$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \,, \quad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2xy \\ x^2 - y^2 \end{bmatrix}$$

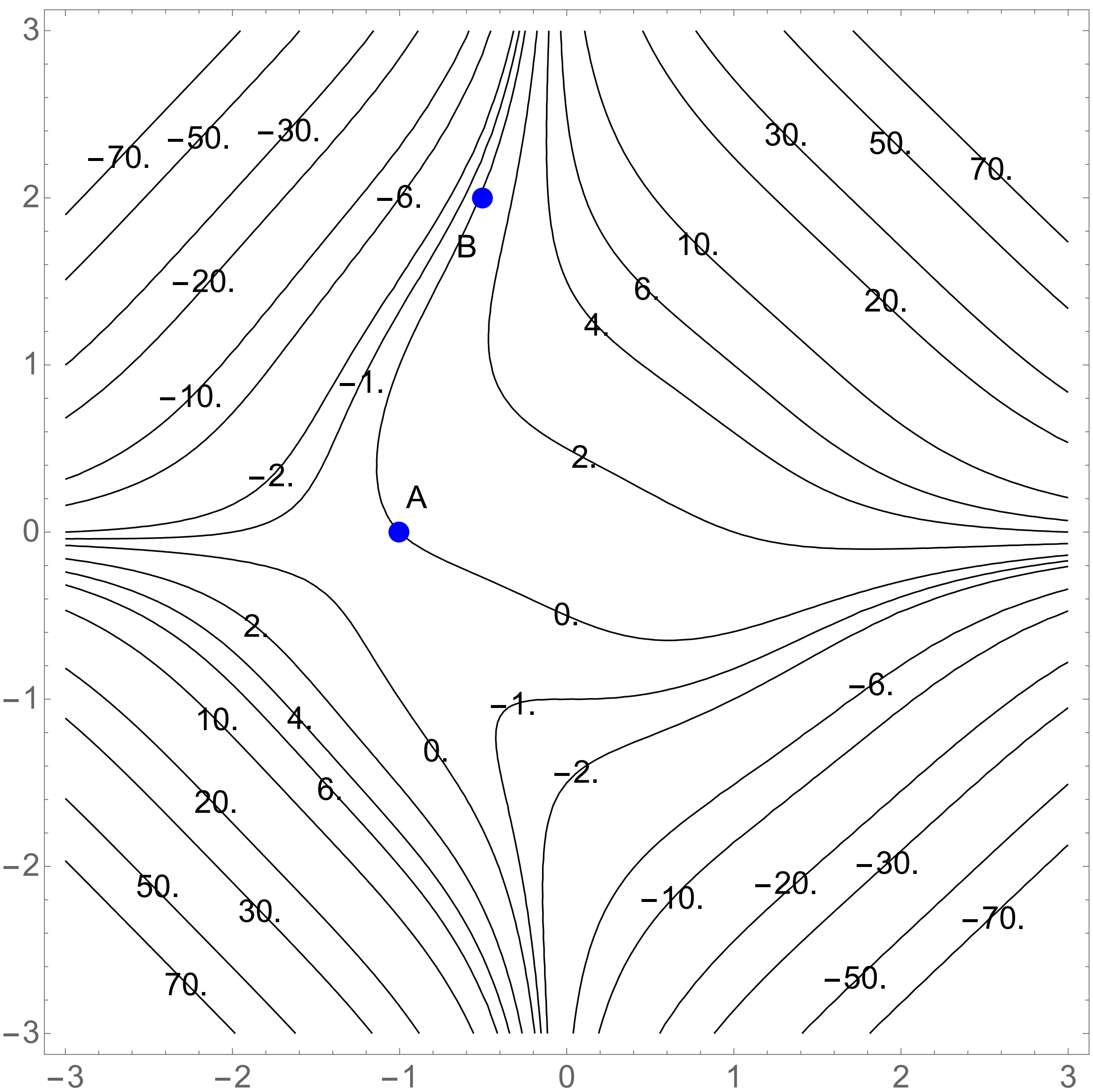
$$h : \mathbb{R}^2 \to \mathbb{R} \,, \ (x,y) \mapsto x^3 + 3xy + y^3$$

(a) Compute the Jacobian matrices Df, Dg, Dh.

(b) Let $p(x,y) = g(y/x, x^2 + y^2)$. Determine Dp.

(c) Let $q(x,y) = h(2xy, x^2 - y^2)$. Determine ∇q and compute $\frac{\partial q}{\partial y}(1,1)$.

5. A function $f(x,y)$ has the following level curve diagram



(a) On the level curve diagram mark the portion(s) of the level curve $f = 2$ where $\frac{\partial f}{\partial y} \geq 0$.

(b) Based on the information given in the level curve diagram, mark the correct relationship between $|\nabla f(A)|$ and $|\nabla f(B)|$:

$\|\nabla f(A)\| < \|\nabla f(B)\|$	$\|\nabla f(A)\| = \|\nabla f(B)\|$	$\|\nabla f(A)\| > \|\nabla f(B)\|$

(c) Let $\underline{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. On the level curve diagram mark a point C satisfying the following conditions:

$$\frac{\partial f}{\partial x}(C) < 0, \quad \text{and} \quad D_{\underline{v}} f(C) > 0$$