Middlebury College

Math 223AB: Spring 2018
Examination I

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Sign the Honor Code Pledge below.
- Write your name on this exam and any extra sheets you hand in.
- You will have 120 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least three** of Problems 2, 3, 4, 5.
- Your final score will be the sum of your score for Problem 1 and the highest possible score obtained from three of the four remaining problems.
- There are **4 blank pages** attached for scratchwork and/or additional space for solutions.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	10 /10
QUESTION 2:	20 /20
QUESTION 3:	20 /20
QUESTION 4:	20 /20
QUESTION 5:	20 /20
TOTAL:	70 /70

NAME: HENRI POINCARÉ

"I have neither given nor received unauthorized aid on this assignment"

1. (10 points) True/False (no justification required)

   (a) For any $\underline{a}, \underline{b} \in \mathbb{R}^3$,

   $$|\underline{a} \times \underline{b}|^2 = |\underline{a}|^2|\underline{b}|^2 - (\underline{a} \cdot \underline{b})^2$$

   Recall that $|\underline{x}| = \sqrt{\underline{x} \cdot \underline{x}}$ is the magnitude of $\underline{x}$.

   (b) The path

   $$\underline{x}(t) = \begin{bmatrix} e^t \\ 2\ln(t) \\ t^{-1} \end{bmatrix}, \quad t > 0,$$

   is a flow line for the vector field $\underline{F} = \begin{bmatrix} x \\ 2z \\ z^2 \end{bmatrix}$.

   (c) If $\underline{x}(t)$ is a path with acceleration $\underline{a}(t)$ then $\underline{x}(t) \times \underline{a}(t)$ is a constant.

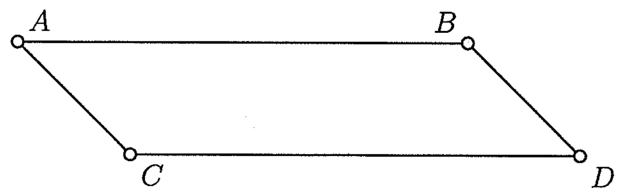
   (d) Let $A, B, C, D \in \mathbb{R}^3$. Then, A, B, C, D all lie in a plane if $(\overrightarrow{AB} \times \overrightarrow{CD}) \cdot \overrightarrow{AC} = 0$

   (e) The line $\underline{r}(t) = \begin{bmatrix} 1+t \\ 2-t \\ 3+5t \end{bmatrix}$ lies in the plane defined by the equation $5x + y = 2$.

Solution: Write T(rue) or F(alse) in the corresponding box below

a)	b)	c)	d)	e)
T	F	F	T	F

2. Let $A = (1,0,2)$, $B = (4,1,2)$, $C = (3,3,3)$, $D = (a,b,c)$



A parallelogram is a planar quadrilateral such that opposite sides are parallel and have equal magnitude.

(a) (3 points) Determine a, b, c so that $ABCD$ is a parallelogram.

(b) (7 points) Determine the equation of the plane containing $ABCD$.

(c) (10 points) Compute the distance between the line segments AC and BD.

a) $\vec{AC} = \vec{BD}$ i.e $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a-4 \\ b-1 \\ c-2 \end{bmatrix} \Rightarrow \begin{matrix} a = 6 \\ b = 4 \\ c = 3 \end{matrix}$

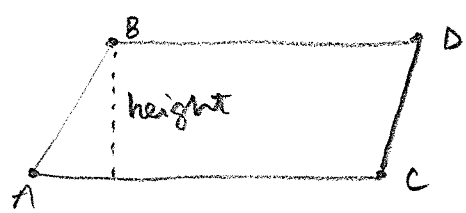
$D = (6,4,3)$

b) $\vec{AB} \times \vec{AC} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} = \vec{n}$

$\vec{n} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{n} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \boxed{x - 3y + 7z = 15}$

c) Note: $|\vec{AC}| = \sqrt{14}$, $|\vec{AB}| = \sqrt{10}$, $\vec{AB} \cdot \vec{AC} = 9 > 0$

$\Rightarrow$



Area = base × height

base = $|\vec{AC}| = \sqrt{14}$

Area = $|\vec{AB} \times \vec{AC}|$

$= \sqrt{59}$

$\Rightarrow$ distance = height

$= \dfrac{\sqrt{59}}{\sqrt{14}}$

3. Let $P = (1,1,0)$, $Q = (0,1,1)$.

(a) (5 points) Determine a parametric description of the line L through P and Q.

(b) (5 points) Given points $A, B \in \mathbb{R}^3$, define the **midpoint between** A **and** B to be the point M such that $\overrightarrow{MA} = -\overrightarrow{MB}$. Determine the midpoint between P and Q.

(c) (5 points) The set of points $R = (x,y,z) \in \mathbb{R}^3$ such that

$$\overrightarrow{PR} \cdot \overrightarrow{PM} = \overrightarrow{QR} \cdot \overrightarrow{QM}$$

defines a plane $\Pi : ax + by + cz = d$. Determine a, b, c, d.

(d) (5 points) Let $R \in \Pi$. Show that $\overrightarrow{MR}$ is perpendicular to L.

a) $\underline{l}(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-t \\ 1 \\ -t \end{bmatrix}$ $t \in \mathbb{R}$.

b) $M = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; $\overrightarrow{MP} = \begin{bmatrix} 1-a \\ 1-b \\ -c \end{bmatrix}$ $\overrightarrow{MQ} = \begin{bmatrix} -a \\ 1-b \\ 1-c \end{bmatrix}$

$\overrightarrow{MP} = -\overrightarrow{MQ} \Rightarrow \begin{bmatrix} 1-a \\ 1-b \\ -c \end{bmatrix} = \begin{bmatrix} a \\ b-1 \\ c-1 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} 1-a = a \\ 1-b = b-1 \\ -c = c-1 \end{array}\right\}$ $\begin{array}{l} a = 1/2 \\ b = 1 \\ c = 1/2 \end{array}$

c) $\overrightarrow{PR} = \begin{bmatrix} x-1 \\ y-1 \\ z \end{bmatrix}$, $\overrightarrow{QR} = \begin{bmatrix} x \\ y-1 \\ z-1 \end{bmatrix}$

$\overrightarrow{PR} \cdot \overrightarrow{PM} = \begin{bmatrix} x-1 \\ y-1 \\ z \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{1}{2}(z+1-x)$

$\overrightarrow{QR} \cdot \overrightarrow{QM} = \begin{bmatrix} x \\ y-1 \\ z-1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \frac{1}{2}(x+1-z)$

$\begin{array}{l} a = 1 \\ b = 0 \\ c = -1 \\ d = 0 \end{array}$

$\Rightarrow z+1-x = x+1-z \Rightarrow \boxed{x = z}$

d) L parallel to normal of Π

$\Rightarrow \overrightarrow{MR} \perp$ to L.

4. Consider the path

$$\underline{x}(t) = \begin{bmatrix} 1/t \\ t \end{bmatrix}, \quad t > 0.$$

Denote by C the image curve of $\underline{x}$.

(a) (5 points) Determine the tangent line to C at $\underline{x}(3)$.

(b) (5 points) Show that there does not exist t_0 for which the line tangent to C at $\underline{x}(t_0)$ is perpendicular to the line tangent to C at $\underline{x}(3)$.

(c) (10 points) Let $0 < t_1 < t_2$. Let L be the line through $\underline{x}(t_1)$ and $\underline{x}(t_2)$. Determine $c > 0$ so that the tangent line to C at $\underline{x}(c)$ is parallel to L.

a) $\underline{x}'(t) = \begin{bmatrix} -1/t^2 \\ 1 \end{bmatrix}$ $\quad \underline{l}(s) = \underline{x}(3) + s\,\underline{x}'(3)$

$= \begin{bmatrix} 1/3 - s/9 \\ 3 + s \end{bmatrix}, \quad s \in \mathbb{R}.$

b) All tangent lines to C have negative slope, since $\underline{x}'(t) = \begin{bmatrix} -1/t^2 \\ 1 \end{bmatrix} \leadsto$ slope $= \dfrac{1}{-1/t^2}$

$= -t^2 < 0$

Since $l(s)$ has negative slope, any line perpendicular to it must have positive slope $\leadsto$ By what we stated above, this is impossible

c) Let $0 < t_1 < t_2$. Want tangent line to be parallel to $\underline{x}(t_2) - \underline{x}(t_1) = \begin{bmatrix} 1/t_2 - 1/t_1 \\ t_2 - t_1 \end{bmatrix} = (t_2 - t_1)\begin{bmatrix} -\frac{1}{t_1 t_2} \\ 1 \end{bmatrix}$

ie require $c > 0$ so that

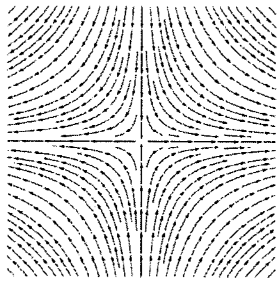
$\underline{x}'(c) = \begin{bmatrix} -1/c^2 \\ 1 \end{bmatrix}$ parallel to $\underline{x}(t_2) - \underline{x}(t_1)$

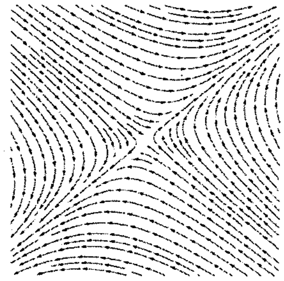
choose $c = \sqrt{t_1 t_2}$. Then, $\underline{x}'(c) = \begin{bmatrix} -1/t_1 t_2 \\ 1 \end{bmatrix}$

is parallel to L.

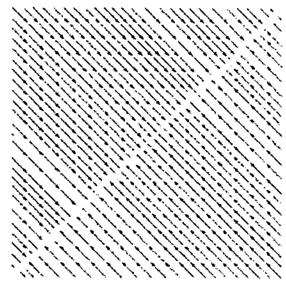
5. (a) (10 points) Match the vector field with the corresponding plot.

Vector Field	$\underline{F} = \begin{bmatrix} x-y \\ y+x \end{bmatrix}$	$\underline{F} = \begin{bmatrix} -x+y \\ 2x-y \end{bmatrix}$	$\underline{F} = \begin{bmatrix} -x+y \\ -y+x \end{bmatrix}$	$\underline{F} = \begin{bmatrix} x \\ -y \end{bmatrix}$
Plot	B	C	D	A


(A)


(B)


(C)


(D)

(b) (10 points) Determine A, B, C, D so that $\underline{x}(t) = \begin{bmatrix} -t+1 \\ 2At^2+Bt+C \\ 5t+D \end{bmatrix}$ is the flow line for the vector field $\underline{F} = \begin{bmatrix} -1 \\ 2x+z \\ 5 \end{bmatrix}$ satisfying $\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underline{x}(0) = \begin{bmatrix} 1 \\ C \\ D \end{bmatrix} \Rightarrow C = D = 1$$

Also:

$$\underline{x}'(t) = \begin{bmatrix} -1 \\ 4At+B \\ 5 \end{bmatrix}$$

$$\underline{F}(\underline{x}(t)) = \begin{bmatrix} -1 \\ -2t+2+5t+1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3t+3 \\ 5 \end{bmatrix}$$

$\Rightarrow \quad 4At + B = 3t+3$

$\Rightarrow \quad A = \frac{3}{4} \quad B = 3.$