MATH 223: Practice Exam 1 Solutions

1a) F $\quad \underline{i} \times ((-\underline{j} \cdot \underline{j})\, \underline{j})$

$$= \underline{i} \times (-\underline{j}) = -\underline{i} \times \underline{j} = -\underline{k}$$

b) F $\quad \underline{x}'(t) = \begin{bmatrix} \cos(t) - t\sin(t) \\ \sin(t) + t\cos(t) \end{bmatrix}$

$$\underline{x}'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

c) F $\quad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0 \Rightarrow$ line parallel to Π

d) F $\quad \underline{x}'(t) = \begin{bmatrix} e^t\cos(t) - e^t\sin(t) \\ -\cos(t) \end{bmatrix}$

$$\underline{F}(\underline{x}(t)) = \begin{bmatrix} e^t\cos(t) + \sin(t) \\ -\sin(t) - e^t\cos(t) \end{bmatrix}$$

$$\neq \underline{x}'(t)$$

e) F

- Normal to plane $2x + y - z = 2$ is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$
- Normal to $\underline{r}(s,t)$ is perp. to $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

ie parallel to

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

not parallel

2) a) $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}$ $\overrightarrow{AD} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $\overrightarrow{BD} = \begin{bmatrix} -1 \\ 1 \\ \sqrt{2} \end{bmatrix}$

θ_1 = angle btwn $\overrightarrow{AB}$, $\overrightarrow{AD}$

$$\cos\theta_1 = \frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}| \cdot |\overrightarrow{AD}|} = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta_1 = \frac{\pi}{3}$$

θ_2 = angle between $\overrightarrow{BA} = -\overrightarrow{AB}$ and $\overrightarrow{BD}$

$$\cos\theta_2 = \frac{\overrightarrow{BA} \cdot \overrightarrow{BD}}{|\overrightarrow{BA}| \cdot |\overrightarrow{BD}|} = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta_2 = \pi/3$$

θ_3 = angle between $\overrightarrow{DB} = -\overrightarrow{BD}$ and $\overrightarrow{DA} = -\overrightarrow{AD}$

$$\cos\theta_3 = \frac{\overrightarrow{DB} \cdot \overrightarrow{DA}}{|\overrightarrow{DB}||\overrightarrow{DA}|} = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta_3 = \pi/3.$$

b) Area ABD $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AD}|$

$$= \frac{1}{2} \left| \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right|$$

$$= \frac{1}{2} \left\| \begin{bmatrix} 2\sqrt{2} \\ 0 \\ 2 \end{bmatrix} \right\| = \sqrt{2+1} = \sqrt{3}$$

Surface area $= 4\sqrt{3}$.

c) $\underline{n}_1 = \dfrac{\overrightarrow{AB} \times \overrightarrow{AD}}{|\overrightarrow{AB} \times \overrightarrow{AD}|}$, $\underline{n}_2 = \dfrac{\overrightarrow{BC} \times \overrightarrow{BD}}{|\overrightarrow{BC} \times \overrightarrow{BD}|}$

$\underline{n}_3 = \dfrac{\overrightarrow{AD} \times \overrightarrow{AC}}{|\overrightarrow{AD} \times \overrightarrow{AC}|}$ $\underline{n}_4 = \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$.

$\Rightarrow \underline{n}_1 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$, $\underline{n}_2 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ \sqrt{2} \\ -1 \end{bmatrix}$

$\underline{n}_3 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$ $\underline{n}_4 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -\sqrt{2} \\ -1 \end{bmatrix}$

<u>Check</u>. $\underline{n}_1 + \underline{n}_2 + \underline{n}_3 + \underline{n}_4 = 0$.

3a) normal vector $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \underline{n}$

Let $\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then,

$$\underline{n} \cdot \underline{x} = \underline{n} \cdot \overrightarrow{OP}$$

$$\Rightarrow \boxed{2x + 3y - z = 4}$$

<u>Note</u>: the computation here is harder than anticipated - not good rep^n of exam!

3b) • Q not on plane : does not satisfy $2x+3y-z=4$

• Distance from P to Π

$\vec{PQ} = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}$ $= \dfrac{\left|\vec{PQ} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right|}{\left\|\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right\|} = \dfrac{22}{\sqrt{14}}$

• $|\vec{QQ'}| = \dfrac{44}{\sqrt{14}} \rightarrow |\vec{QQ'}|^2 = \dfrac{44^2}{14}$

• $\vec{QQ'} = \begin{bmatrix} a-4 \\ b-5 \\ c+3 \end{bmatrix}$; $\vec{QQ'}$ parallel to $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

ie $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{QQ'} \times \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -b-3c-4 \\ a+2c+2 \\ 3a+2b-2 \end{bmatrix}$

ie
$$\left.\begin{aligned} -b-3c &= 4 \\ a \quad +2c &= -2 \\ 3a \quad +2b \quad &= 2 \end{aligned}\right\}$$
solve
$a+2c=-2$
$b+3c=4$

Then, $|\vec{QQ'}|^2 = \vec{QQ'} \cdot \vec{QQ'}$

$= \begin{bmatrix} -6-2c \\ -1-3c \\ c+3 \end{bmatrix} \cdot \begin{bmatrix} -6-2c \\ -1-3c \\ c+3 \end{bmatrix}$

$= (6+2c)^2 + (1+3c)^2 + (c+3)^2$

ie $\dfrac{44^2}{14} = 14c^2 + 36c + 46$

$\Rightarrow \quad 0 = 196c^2 + 504c - 1292$

$= 4(49c^2 + 126c - 323)$

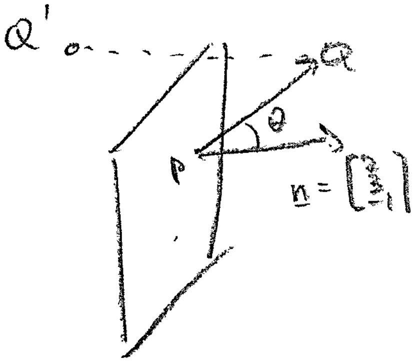
$$c = \frac{-126 \pm \sqrt{126^2 + 4 \cdot 49 \cdot 323}}{98}$$

$= -1.2857.. \pm 2.87139..$

Since $\overrightarrow{PQ} = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}$ and $\overrightarrow{PQ} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} > 0$

$\Rightarrow$ angle btwn $\overrightarrow{PQ}$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} < \pi/2$

Want; angle between $\overrightarrow{PQ'}$ and $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ to be $> \pi/2$



ie $\overrightarrow{PQ'} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} < 0$

$\overrightarrow{PQ'} = \begin{bmatrix} a-1 \\ b-1 \\ c-1 \end{bmatrix} = \begin{bmatrix} -3-2c \\ 3-3c \\ c-1 \end{bmatrix}$

Hence, $\overrightarrow{PQ'} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = -6-4c+9-9c-c+1$

$= 4 - 14c < 0$

$\Rightarrow \quad \frac{4}{14} < c$

Take $c = -1.2857 + 2.87139..$

Then; $Q' = (a, b, c) = (-2-2c, 4-3c, c)$

4a) L represented with parametric equation

$$\underline{r}(s) = \underline{x}(\pi/4) + s\,\underline{x}'(\pi/4)$$

$$= \begin{bmatrix} 3/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} + s \begin{bmatrix} -3/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{2} - s\,3/\sqrt{2} \\ 2/\sqrt{2} + s\,2/\sqrt{2} \end{bmatrix}$$

b) L intersects when $\underline{r}(s) = \begin{bmatrix} 0 \\ a \end{bmatrix}$

ie $\frac{3}{\sqrt{2}}(1-s) = 0$

$\Rightarrow s = 1$

Hence, $\underline{r}(1) = \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} = P.$

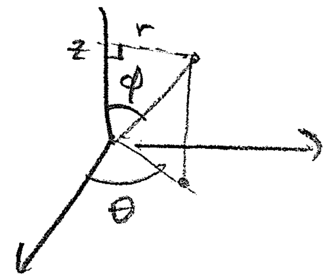
c) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \begin{bmatrix} -3/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 0 \\ 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ -\frac{2}{\sqrt{2}} \end{bmatrix}$$

$\parallel$ ✓

$$\underline{x}'(t) = \begin{bmatrix} -3\sin(t) \\ 2\cos(t) \end{bmatrix} \Rightarrow \underline{x}'\left(\frac{3\pi}{4}\right) = \begin{bmatrix} -3/\sqrt{2} \\ -2/\sqrt{2} \end{bmatrix}.$$

5a) $r^2 = x^2 + y^2 = z^2 \tan^2 \varphi$



$\Rightarrow \quad r = z \tan \varphi$

$\theta = \theta$

$z = z$

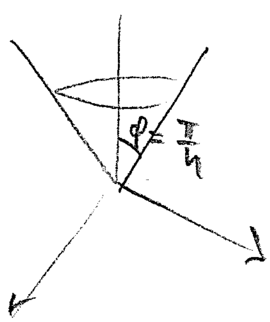
b) $x^2 + y^2 = z^2$

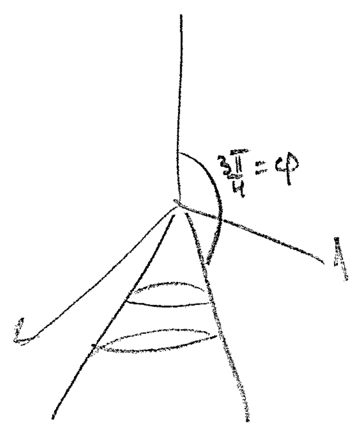
$z^2 \tan^2 \varphi = z^2$

$\Rightarrow \quad \tan^2 \varphi = 1$

$\Rightarrow \quad \tan \varphi = \pm 1$

$\Rightarrow \quad \varphi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.





c) $z = \cos \varphi \Rightarrow z^2 = \cos^2 \varphi$

$$x^2 + y^2 + z^2 = z^2 \tan^2 \varphi + z^2$$
$$= \cos^2 \varphi \tan^2 \varphi + \cos^2 \varphi$$
$$= \sin^2 \varphi + \cos^2 \varphi = 1$$

$\Rightarrow$ Sphere of radius 1 centre at O.