Middlebury College

Math 223AB: Spring 2018
Contact: `gmelvin@middlebury.edu`

Practice Examination I

Instructions:

- You *must* attempt Problem 1.
- Please attempt at least three of Problems 2,3,4,5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the scores for the three remaining problems receiving the highest number points.
- Calculators are not permitted.

---

1. (10 points) True/False:

   (a) The vector $\left(\underline{i} \times \left(\left(-\underline{j} \cdot \underline{j}\right) \underline{j}\right)\right) \times \underline{j}$ points in the direction of $\underline{k}$.

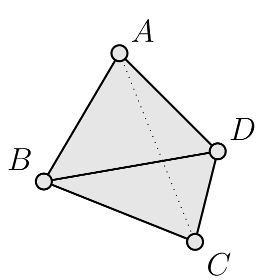
   (b) The slope of the tangent line to the image curve of $\underline{x}(t) = \begin{bmatrix} t\cos(t) \\ t\sin(t) \end{bmatrix}$, $t \in \mathbb{R}$, at $\underline{x}(0)$ is undefined.

   (c) Let Π be the plane with normal vector $\overrightarrow{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ containing the point $P = (0, 1, -1)$. Then, the line through $Q = (3, 2, -5)$ in the direction $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ intersects Π.

   (d) Consider the vector field $\underline{F} = \begin{bmatrix} x + y \\ y - x \end{bmatrix}$ in the plane. The path $\underline{x}(t) = \begin{bmatrix} e^t \cos(t) \\ -\sin(t) \end{bmatrix}$, $\underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, is a flow line for $\underline{F}$.

   (e) The plane defined by the equation $2x + y - z = 2$ is parallel to the plane $\underline{r}(s,t) = \begin{bmatrix} 1 + s - t \\ 2s \\ 3 - s + 2t \end{bmatrix}$.

2. The points $A = (0, -1, 1/\sqrt{2})$, $B = (1, 0, -1/\sqrt{2})$, $C = (-1, 0, -1/\sqrt{2})$, $D = (0, 1, 1/\sqrt{2})$ are the vertices of a **tetrahedron** T.



A tetrahedron is characterised by the property that any three vertices define an equilateral triangle. In particular, all of its edges have the same length.

(a) Verify that the triangle ABD is equilateral i.e. show that each of the interior angles is $\pi/3$.

(b) Determine the area of ABD. Determine the surface area of the tetrahedron.

(c) Let $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, $\vec{n}_4$ be outward-pointing, unit length normal vectors to each of the faces of T. Show that

$$\vec{n}_1 + \vec{n}_2 + \vec{n}_3 + \vec{n}_4 = \vec{0}.$$

3. (a) Determine the plane Π that's perpendicular to the line $\underline{r}(t) = \begin{bmatrix} -1+2t \\ 3t \\ 2-t \end{bmatrix}$, $t \in \mathbb{R}$, and containing the point $P = (1,1,1)$.

   (b) Let $Q = (4,5,-3)$. Define $Q' = (a,b,c)$ to be the point that is the *mirror image* of Q on the opposite side of the plane: this means $\overrightarrow{QQ'}$ is normal to Π and $|\overrightarrow{QQ'}|$ is twice the distance from Q to Π. Determine Q'.

4. Consider the path

$$\underline{x}(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) \end{bmatrix}, \quad t \in \mathbb{R}$$

   Denote by C the image curve of $\underline{x}$.

   (a) Let L denote the tangent line to C at $\underline{x}(\pi/4)$. Determine a parametric equation for L.

   (b) L intersects the y-axis at a unique point $P = (0,a)$. Determine a.

   (c) Let $Q = \underline{x}(3\pi/4) \in C$. Show that $\overrightarrow{PQ} = \underline{x}'(3\pi/4)$, the velocity vector of $\underline{x}$ at $t = 3\pi/4$.

5. Define *spheryndrical coordinates* (φ, θ, z), $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$, $z \in \mathbb{R}$, to be the coordinate system related to Cartesian coordinates via the transformation

$$\begin{aligned} x &= z\tan\varphi\cos\theta \\ y &= z\tan\varphi\sin\theta \\ z &= z \end{aligned}$$

   (a) Write down the cylindrical coordinates in terms of spheryndrical coordinates

$$r =$$

$$\theta =$$

$$z =$$

   Recall that $r \ge 0$, $0 \le \theta \le 2\pi$, $z \in \mathbb{R}$.

   (b) Write down a description in terms of spheryndrical coordinates of the double-napped cone $x^2 + y^2 = z^2$.

   (c) Describe the surface given by the spheryndrical equation $z = \cos\varphi$.