



APRIL 9 LECTURE

THE POTENTIAL FUNCTION PROBLEM

LEARNING OBJECTIVES:

- Understand the technique of differentiation under the integral.
- Understand what it means for a vector field to admit local potential functions.

Let $\underline{F} = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$ be a vector field on \mathbb{R}^2 . We are going to investigate the following multivariable calculus analog of the antiderivative problem:

The Potential Function Problem:

Can we find a function f so that $\underline{F} = \nabla f$?

That is, we want to determine the existence of a potential function f for \underline{F} .

Over the next few lectures we will determine (completely) the solution to this existence problem. We will need to introduce some technical results and new techniques:

- differentiation under the integral
- line integrals

Differentiation under the integral

Consider the function $f(x, y) = \sin(xy)$, with domain \mathbb{R}^2 . Define the function

$$\psi(y) = \int_0^\pi \sin(xy) dx$$

As a function of y , we see that

$$\psi(y) = \left[-\frac{1}{y} \cos(xy) \right]_0^\pi = \frac{1}{y} (1 - \cos(\pi y))$$

Differentiating with respect to y gives

$$\psi'(y) = -\frac{1}{y^2} (1 - \cos(\pi y)) + \pi \sin(\pi y)$$

MATHEMATICAL WORKOUT

Using integration by parts, compute

$$\int_0^\pi \frac{\partial f}{\partial y} dx$$

Recall that the method of integration by parts states $\int_a^b f g' = [f g]_a^b - \int_a^b f' g$.

$$\begin{aligned}
 f &= x & f' &= 1 \\
 g' &= \cos(xy) & g &= \frac{\sin(xy)}{y}
 \end{aligned}$$

$$\begin{aligned}
 \int_0^\pi x \cos(xy) dx &= \left[\frac{x}{y} \sin(xy) \right]_0^\pi - \int_0^\pi \frac{\sin(xy)}{y} dx \\
 &= \frac{\pi}{y} \sin(\pi y) + \left[\frac{\cos(xy)}{y^2} \right]_0^\pi \\
 &= \frac{\pi}{y} \sin(\pi y) + \frac{\cos(\pi y)}{y^2} - \frac{1}{y^2} = \psi'(y)!
 \end{aligned}$$

This example is an instance of the following result:

Differentiation under the integral sign

Let $f(x, y)$ be a continuous function of two variables defined over the rectangle $a \leq x \leq b, c \leq y \leq d$. Assume $\frac{\partial f}{\partial y}$ is continuous. If

$$\psi(y) = \int_a^b f(x, y) dx$$

then

$$\frac{d\psi}{dy} = \int_a^b \frac{\partial f(x, y)}{\partial y} dx \quad (*)$$

More generally, if

$$\psi(x, y) = \int_a^x f(t, y) dt$$

then

$$\frac{\partial \psi}{\partial y} = \int_a^x \frac{\partial f(t, y)}{\partial y} dt \quad (**)$$

Remark:

1. The proof of this result requires showing that the limit used to define $\frac{d\psi}{dy}$ and the limit used to define the definite integral (i.e. as a limit of Riemann sums) can be interchanged; this requires a careful $\epsilon - \delta$ argument and relies on the assumption that $\frac{\partial f}{\partial y}$ is continuous.
2. The formula (*) is completely different from the differentiation in the Fundamental Theorem of Calculus: the F.T.o.C. states, if

$$g(x) = \int_a^x h(t) dt$$

then $\frac{d}{dx}(g(x)) = h(x)$. In particular, for a function $f(x, y)$, $\psi(x, y)$ defined as above, the F.T.o.C. gives

$$\frac{\partial \psi}{\partial x} = f(x, y)$$

For example, if $f(x, y) = \sin(xy)$ and

$$\psi(x, y) = \int_0^x \sin(ty) dt$$

then the F.T.o.C. gives $\frac{\partial \psi}{\partial x} = \sin(xy)$. However, (**) states that

$$\frac{\partial \psi}{\partial y} = \int_0^x t \cos(ty) dt$$

Example: Define

$$\psi(x, y) = \int_0^x e^{y+t} dt$$

Evaluating the integral directly gives

$$\psi(x, y) = [e^{y+t}]_0^x = e^y(e^x - 1)$$

Therefore, $\frac{\partial \psi}{\partial y} = e^y(e^x - 1) = \psi$. Alternatively, using (**) and differentiating under the integral, we have

$$\frac{\partial \psi}{\partial y} = \int_0^x \frac{\partial}{\partial y} (e^{y+t}) dt = \int_0^x e^{y+t} dt = \psi$$

Local existence of potential functions

Let $\underline{F} : X \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\underline{F}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$, be a vector field on \mathbb{R}^2 . Assume that u, v are differentiable and have continuous partial derivatives.

In this section we will show that the Potential Function Problem can be solved whenever X is the entire plane, an open disc (a set of the form $\{(x, y) \mid (x-a)^2 + (y-b)^2 < r^2\}$), or an open rectangle $a < x < b$, $c < y < d$, and

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Suppose that X is the entire plane - the proof for the other cases is similar - and that $u_y = v_x$. Fix $(a, b) \in X$. For $(x, y) \in X$, define

$$f(x, y) = \int_a^x u(s, y) ds + \int_b^y v(a, t) dt$$

We claim: $\nabla f = \underline{F}$. We compute

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(\int_a^x u(s, y) ds + \int_b^y v(a, t) dt \right) = u(x, y)$$

since the second integral is independent of x . On the other hand,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left(\int_a^x u(s, y) ds + \int_b^y v(a, t) dt \right) \\ &= \frac{\partial}{\partial y} \left(\int_a^x u(s, y) ds \right) + v(a, y), \quad \text{by the F.T.o.C.} \\ &= \int_a^x \frac{\partial u(s, y)}{\partial y} ds + v(a, y), \quad \text{differentiating under the integral} \\ &= \int_a^x \frac{\partial v(s, y)}{\partial s} ds + v(a, y), \quad \text{since } \frac{\partial u(s, y)}{\partial y} = \frac{\partial v(s, y)}{\partial s}, \text{ by assumption} \\ &= [v(s, y)]_a^x + v(a, y) \\ &= v(x, y) - v(a, y) + v(a, y) = v(x, y) \end{aligned}$$

We have just defined a function $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x} = u(x, y), \quad \text{and} \quad \frac{\partial f}{\partial y} = v(x, y) \quad \implies \quad \nabla f = \underline{F}$$

Important Remark: The vector field

$$\underline{F} = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

is defined on $X = \mathbb{R}^2 - \{(0, 0)\}$. Moreover,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

but, as we will see in a few lectures, there does not exist a function f whose domain is X that satisfies $\nabla f = \underline{F}$. However, the result proved above shows that there does exist a function f with (for example) domain $X : 1 < x < 2, 1 < y < 2$ that satisfies $\nabla f = \underline{F}$.