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April 9 Lecture

Gradient vector fields and potential functions

Learning Objectives:
- *Understand the technique of differentiation under the integral.*
- *Understand what it means for a vector field to admit local potential functions.*

Let $\underline{F} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ be a vector field on $\mathbb{R}^2$. We are going to investigate the following multivariable calculus analog of the **antiderivative problem**:

The Potential Function Problem:

Can we find a function f so that $\underline{F} = \nabla f$?

That is, we want to determine the existence of a **potential function** f for $\underline{F}$.

Over the next few lectures we will determine (completely) the solution to this existence problem. We will need to introduce some technical results and new techniques:

- *differentiation under the integral*
- *line integrals*

Differentiation under the integral

Consider the function $f(x,y) = \sin(xy)$, with domain $\mathbb{R}^2$. Define the function

$$\psi(y) = \int_0^{\pi} \sin(xy)dx$$

As a function of y, we see that

$$\psi(y) = \left[-\frac{1}{y}\cos(xy) \right]_0^{\pi} = \frac{1}{y}\left(1 - \cos(\pi y)\right)$$

Differentiating with respect to y gives

$$\psi'(y) = -\frac{1}{y^2}\left(1 - \cos(\pi y) + \pi \sin(\pi y)\right)$$

Mathematical workout
Using integration by parts, compute

$$\int_0^{\pi} \frac{\partial f}{\partial y} dx$$

Recall that the method of integration by parts states $\int_a^b fg' = [fg]_a^b - \int_a^b f'g$.

This example is an instance of the following result:

Differentiation under the integral sign

Let $f(x,y)$ be a continuous function of two variables defined over the rectangle $a \leq x \leq b, c \leq y \leq d$. Assume $\frac{\partial f}{\partial y}$ is continuous. If

$$\psi(y) = \int_a^b f(x,y)dx$$

then

$$\frac{d\psi}{dy} = \int_a^b \frac{\partial f(x,y)}{\partial y} dx \qquad (*)$$

More generally, if

$$\psi(x,y) = \int_a^x f(t,y)dt$$

then

$$\frac{\partial \psi}{\partial y} = \int_a^x \frac{\partial f(t,y)}{\partial y} dt \qquad (**)$$

Remark:

1. The proof of this result requires showing that the limit used to define $\frac{d\psi}{dy}$ and the limit used to define the definite integral (i.e. as a limit of Riemann sums) can be interchanged; this requires a careful $\epsilon - \delta$ argument and relies on the assumption that $\frac{\partial f}{\partial y}$ is continuous.

2. The formula $(*)$ is **completely different from the differentiation in the Fundamental Theorem of Calculus**: the F.T.o.C. states, if

$$g(x) = \int_a^x h(t)dt$$

then $\frac{d}{dx}(g(x)) = h(x)$. In particular, for a function $f(x,y)$, $\psi(x,y)$ defined as above, the F.T.o.C. gives

$$\frac{\partial \psi}{\partial x} = f(x,y)$$

For example, if $f(x,y) = \sin(xy)$ and

$$\psi(x,y) = \int_0^x \sin(ty)dt$$

then the F.T.o.C. gives $\frac{\partial \psi}{\partial x} = \sin(xy)$. However, $(**)$ states that

$$\frac{\partial \psi}{\partial y} = \int_0^x t\cos(ty)dt$$

Example: Define

$$\psi(x,y) = \int_0^x e^{y+t}dt$$

Evaluating the integral directly gives

$$\psi(x,y) = \left[e^{y+t}\right]_0^x = e^y(e^x - 1)$$

Therefore, $\frac{\partial \psi}{\partial y} = e^y(e^x-1) = \psi$. Alternatively, using $(**)$ and differentiating under the integral, we have

$$\frac{\partial \psi}{\partial y} = \int_0^x \frac{\partial}{\partial y}(e^{y+t})dt = \int_0^x e^{y+t}dt = \psi$$

Local existence of potential functions

Let $\underline{F} : X \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\underline{F}(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$, be a vector field on $\mathbb{R}^2$. Assume that u, v are differentiable and have continuous partial derivatives.

In this section we will show that the Potential Function Problem can be solved **whenever X is the entire plane, an open disc (a set of the form $\{(x,y) \mid (x-a)^2 + (y-b)^2 < r^2\}$), or an open rectangle $a < x < b$, $c < y < d$,** and

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

Suppose that X is the entire plane - the proof for the other cases is similar - and that $u_y = v_x$. Fix $(a,b) \in X$. For $(x,y) \in X$, define

$$f(x,y) = \int_a^x u(s,y)ds + \int_b^y v(a,t)dt$$

We claim: $\nabla f = \underline{F}$. We compute

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_a^x u(s,y)ds + \int_b^y v(a,t)dt\right) = u(x,y)$$

since the second integral is independent of x. On the other hand,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\int_a^x u(s,y)ds + \int_b^y v(a,t)dt\right) \\
&= \frac{\partial}{\partial y}\left(\int_a^x u(s,y)ds\right) + v(a,y), \quad \text{by the F.T.o.C.} \\
&= \int_a^x \frac{\partial u(s,y)}{\partial y}ds + v(a,y), \quad \text{differentiating under the integral} \\
&= \int_a^x \frac{\partial v(s,y)}{\partial s}ds + v(a,y), \quad \text{since } \frac{\partial u(s,y)}{\partial y} = \frac{\partial v(s,y)}{\partial s}, \text{ by assumption} \\
&= [v(s,y)]_a^x + v(a,y) \\
&= v(x,y) - v(a,y) + v(a,y) = v(x,y)
\end{aligned}$$

We have just defined a function $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x} = u(x, y), \quad \text{and} \quad \frac{\partial f}{\partial y} = v(x, y) \qquad \Longrightarrow \qquad \nabla f = \underline{F}$$

Important Remark: The vector field

$$\underline{F} = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

is defined on $X = \mathbb{R}^2 - \{(0, 0)\}$. Moreover,

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

but, as we will see in a few lectures, **there does not exist a function f whose domain is X that satisfies $\nabla f = \underline{F}$.** However, the result proved above shows that there does exist a function f with (for example) domain $X : 1 < x < 2, 1 < y < 2$ that satisfies $\nabla f = \underline{F}$.