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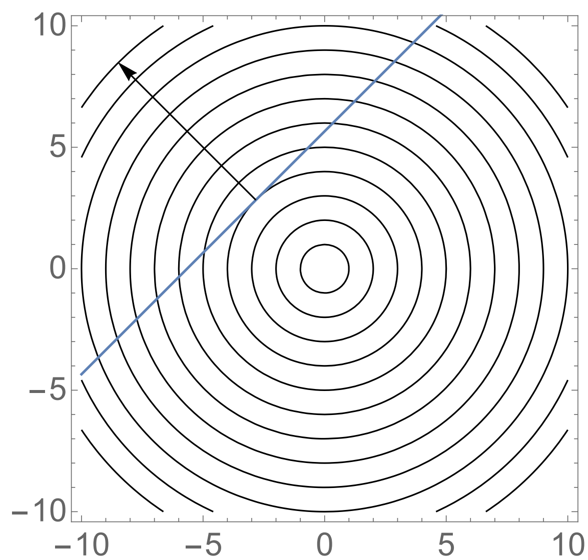
April 6 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §3.3, 6.3
- *Calculus of Several Variables*, Lang: §V.1

Gradient vector fields and potential functions

Learning Objectives:
- *Learn what it means for a vector field to be conservative.*
- *Learn the definition of a potential function.*
- *Learn how to find potential functions for certain conservative vector fields.*
- *Learn the Test for Non-conservative Vector Fields*

Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a differentiable scalar-valued function. Then, for $\underline{a} \in X$, $\nabla f(\underline{a})$ is orthogonal to the tangent line of the level curve of f passing through $\underline{a}$.



For example, the above level curve diagram is for $f(x,y) = x^2 + y^2$, with tangent line to the level curve

$$x^2 + y^2 = 16$$

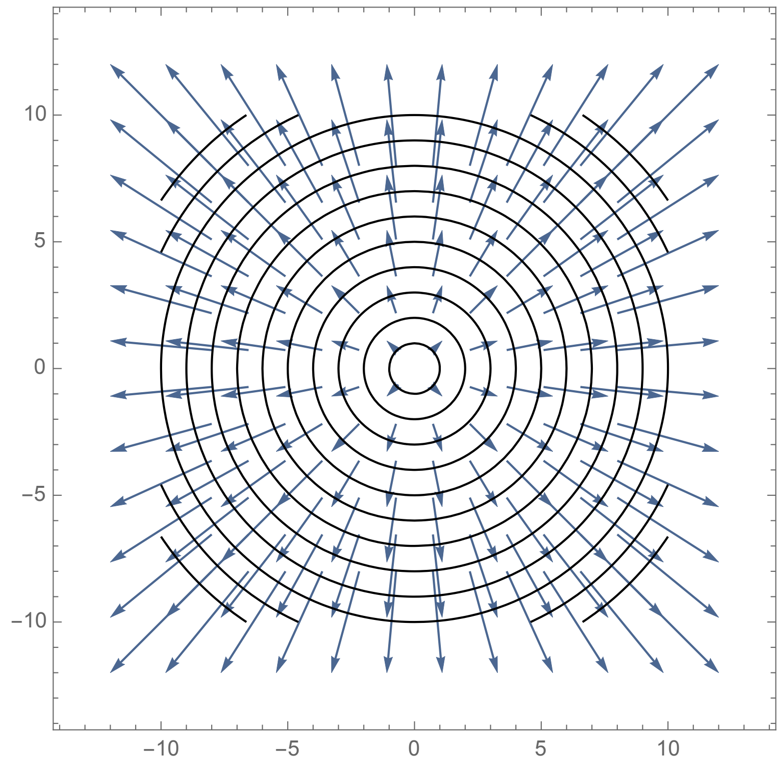
at the point $(-\sqrt{8}, \sqrt{8})$ and gradient $\nabla f(-\sqrt{8}, \sqrt{8})$ indicated. More generally, the gradient is

$$\nabla f = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

Writing the transpose of this 1×2 matrix gives a **vector field**

$$\underline{F}(x,y) = (\nabla f)^t = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

The level curves of f are **orthogonal** to the vector field



In particular, **the level curves of f are orthogonal to the flow lines of $\underline{F}$:** the level curves are **NOT** flow lines!

> **Problem:**
>
> Given any vector field $\underline{F}$ can we find a function f so that
> $$\underline{F} = (\nabla f)^t?$$

Remark: By an abuse of notation, we will now consider ∇f as a column vector, so we don't have to keep writing the more cumbersome $(\nabla f)^t$.

Example:

1. Consider the vector field $\underline{F} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. If $f(x, y)$ is a function satisfying

   $$\nabla f = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \Longrightarrow \quad \frac{\partial f}{\partial x} = 2, \quad \text{and} \quad \frac{\partial f}{\partial y} = -3$$

   Hence,

   $$\frac{\partial f}{\partial x} = 2 \quad \Longrightarrow f(x, y) = 2x + g(y)$$

   Notice that our '*constant of integration*' must be a function of y, given how partial differentiation (with respect to x) is defined. Thus,

   $$-3 = \frac{\partial f}{\partial y} = \frac{dg}{dy} \quad \Longrightarrow \quad g(y) = -3y + C$$

   where C is a constant.

   Then, if we let $f(x, y) = 2x - 3y + 10$, for example, then $\nabla f = \underline{F}$. Observe that the constant '+10' is arbitrary and could be changed to any constant.

2. Consider the vector field $\underline{F} = \begin{bmatrix} 2x^3y^4 + x \\ 2x^4y^3 + y \end{bmatrix}$. If $\nabla h = \underline{F}$, for some $h(x, y)$, then

$$\frac{\partial h}{\partial x} = 2x^3y^4 + x, \quad \text{and} \quad \frac{\partial h}{\partial y} = 2x^4y^3 + y$$

Integrating with respect to x, and recalling that we consider y to be constant when taking the partial derivative with respect to x,

$$\frac{\partial h}{\partial x} = 2x^3y^4 + x \quad \implies \quad h(x, y) = \frac{x^4y^4}{2} + \frac{x^2}{2} + k(y)$$

and

$$2x^4y^3 + y = \frac{\partial h}{\partial y} = 2x^4y^3 + k'(y) \quad \implies \quad k'(y) = y$$

Hence,

$$k(y) = \int y dy = \frac{y^2}{2} + C$$

where C is a constant of integration. Therefore, if we let

$$h(x, y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1$$

Then, $\nabla h = \underline{F}$. Observe that the constant '+1' is arbitrary and could be changed to any constant.

Let $\underline{F}$ be a vector field. If there exists a function f so that $\nabla f = \underline{F}$ then we say that $\underline{F}$ is a **conservative (or gradient) vector field**, and call f a **potential function** of $\underline{F}$.

Example: Both vector fields given in the Example above are conservative. The functions $f(x, y) = 2x - 3y + 10$ and $h(x, y) = \frac{1}{2}(x^4y^4 + x^2 + y^2) + 1$ are potential functions.

Our **Problem** above is translated to: let $\underline{F}$ be a vector field. Is $\underline{F}$ conservative?

> **Problem:**
>
> Let $\underline{F}$ be a vector field. Is $\underline{F}$ conservative?

Remark: the above problem can be posed for a vector field $\underline{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ defined on $\mathbb{R}^n$.

We will now see a useful criterion for showing a vector field is **not** conservative in the case that $\underline{F}$ is a vector field in $\mathbb{R}^2$.

Suppose that $\underline{F} = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$ is conservative, say $\underline{F} = \nabla f$. Therefore,

$$u(x, y) = \frac{\partial f}{\partial x}, \quad \text{and} \quad v(x, y) = \frac{\partial f}{\partial y}$$

If we assume that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable and have continuous partial derivatives then **Clairaut's Theorem** (see March 21 Lecture) states that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

In particular, if $\underline{F} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is conservative then

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

Therefore,

Test for Non-conservative Vector Fields

Let

$$\underline{F} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$$

be a vector field on $\mathbb{R}^2$. If

$$\frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$$

then $\underline{F}$ is **not** conservative.

Example: Consider the vector field

$$\underline{F} = \begin{bmatrix} x^2 y \\ \sin(xy) \end{bmatrix}$$

Here

$$u(x,y) = x^2 y, \quad \text{and} \quad v(x,y) = \sin(xy)$$

Both u and v are differentiable with continuous partial derivatives. We compute

$$\frac{\partial u}{\partial y} = x^2 \neq y\cos(xy) = \frac{\partial v}{\partial x}$$

Hence, $\underline{F}$ is not conservative.

Important Remark: The vector field

$$\underline{F} = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

defined on $\mathbb{R}^2 - \{(0,0)\}$, satisfies (exercise!)

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

but we will soon see that $\underline{F}$ is **not** conservative!