Middlebury College

Multivariable Calculus
Spring 2018
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April 4 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §2.6

Directional Derivative

Learning Objectives:
- *Learn how to compute the directional derivative.*
- *Learn how to compute the tangent line/plane to a level curve/surface.*

Directional Derivatives: Given a differentiable function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $\underline{a} \in X$, $\underline{v} \in \mathbb{R}^n$, the **directional derivative of f at $\underline{a}$ in the direction $\underline{v}$** is

$$D_{\underline{v}} f(\underline{a}) = \nabla f(\underline{a}) \underline{v}$$

1. Compute the directional derivative of $f(x,y) = x^2 + 3xy + y^2$ at the point $(2,1)$ in the direction that points towards the origin.

- $\nabla f = \begin{bmatrix} 2x+3y & 3x+2y \end{bmatrix} \Rightarrow \nabla f(2,1) = \begin{bmatrix} 7 & 8 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$\Rightarrow D_{\underline{v}} f(2,1) = \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \underline{\underline{-22}}$

2. Find a nonzero vector $\underline{v} \in \mathbb{R}^2$ so that $D_{\underline{v}} f(2,1) = 0$

Let $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then, require

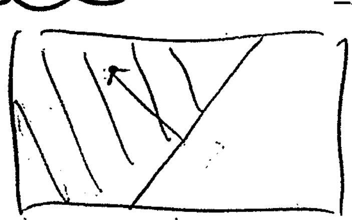
$0 = D_{\underline{v}} f(2,1) = \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 7a + 8b$

eg $\underline{v} = \begin{bmatrix} -8 \\ 7 \end{bmatrix}$

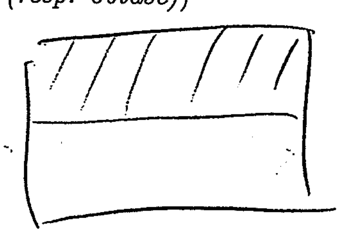
ALSO: $\underline{w} = \begin{bmatrix} x \\ y \end{bmatrix}$, $y > 0$,

$\nabla f(2,1) \underline{w} > 0$

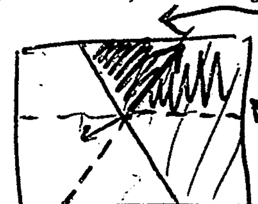
3. Let $\underline{w} \in \mathbb{R}^2$ be a vector satisfying $\underline{w} \cdot \underline{v} > 0$ and $\underline{w} \cdot \begin{bmatrix} -2 \\ -1 \end{bmatrix} < 0$. Is f increasing in the direction $\underline{w}$ at $(2,1)$? (*Recall that, if $\underline{a} \cdot \underline{b} > 0$ (resp. < 0) then angle between $\underline{a}$, $\underline{b}$ is acute (resp. obtuse)*)



$\underline{w} \cdot \underline{v} > 0$



$y > 0$



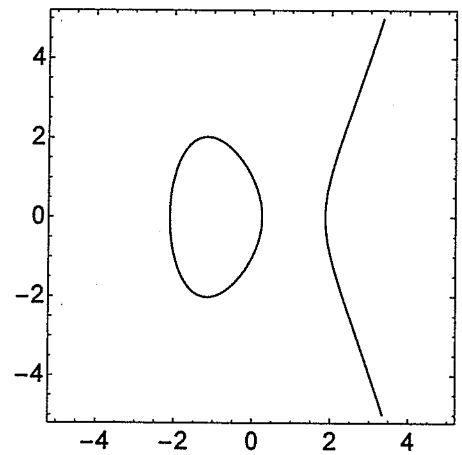
$\underline{w}$ points here or here

$\underline{w} \cdot \begin{bmatrix} -2 \\ -1 \end{bmatrix} < 0$ $\underline{\underline{\text{YES}}}$

Tangent lines/planes of level sets: Let S be a level set of $f : X \subset \mathbb{R}^n \to \mathbb{R}$, where $n = 2, 3$. This means $S = \{\underline{x} \mid f(\underline{x}) = c\}$, for some c. Then, $\nabla f(\underline{a})$ is

- perpendicular to the tangent line to S at $\underline{a}$ ($n = 2$),
- normal to the tangent plane to S at $\underline{a}$ ($n = 3$).

1. Consider the curve $y^2 = x^3 - 4x + 1$.



(a) Find a function $h(x, y)$ so that the curve is a level curve of h

$$h(x,y) = y^2 - x^3 + 4x - 1$$

(b) Compute the tangent line to the curve at $(2, 1)$.

$$\nabla h = \begin{bmatrix} -3x^2 + 4 & 2y \end{bmatrix}$$

$$\nabla h(2,1) = \begin{bmatrix} -8 & 2 \end{bmatrix}$$

Tangent line $\perp$ $\nabla h(2,1)$ $\Rightarrow$ slope of tangent line is 4

$$y - 1 = 4(x - 2) \Rightarrow y = 4x - 7$$

(c) Determine the point (x, y), $y > 0$, where the tangent line is horizontal.

Require $\nabla h(x,y)$ to be vertical

$$\Rightarrow \quad 4 - 3x^2 = 0 \quad \Rightarrow \quad x = \pm \frac{2}{\sqrt{3}}$$

Note: • If $x = \frac{2}{\sqrt{3}}$, $x^3 - 4x + 1 = \frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}} + 1 = 1 - \frac{16}{\sqrt{3}} < 0$

ie no y-value so that $\left(\frac{2}{\sqrt{3}}, y\right)$ on curve.

• $x = \frac{-2}{\sqrt{3}} \Rightarrow y^2 = 1 + \frac{16}{\sqrt{3}} \Rightarrow y = \sqrt{1 + \frac{16}{\sqrt{3}}}$

Hence, horizontal tangent line @ $\left(\frac{-2}{\sqrt{3}}, \sqrt{1 + \frac{16}{\sqrt{3}}}\right)$.

2. Consider the paraboloid $z = x^2+y^2$. This is the graph of the function $f(x, y) = x^2 + y^2$.

(a) Find a function $g(x, y, z)$ so that the paraboloid is a level set of g.

$$g(x,y,z) = z - x^2 - y^2$$

(b) Determine the tangent plane to the paraboloid at $(1, 2, 5)$

$$\nabla g = \begin{bmatrix} -2x & -2y & 1 \end{bmatrix}$$

$$\Rightarrow \nabla g(1,2,5) = \begin{bmatrix} -2 & -4 & 1 \end{bmatrix} \leftarrow$$

normal to tgt plane

$\Rightarrow$ tangent plane is

$$-2(x-1) - 4(y-2) + (z-5) = 0$$

$$\Rightarrow -2x - 4y + z = 5$$

(c) Suppose that $z = f(x, y)$ is a surface. Generalise your approach above to determine the tangent plane to the surface at $(a, b, f(a, b))$.

Let $g(x,y,z) = z - f(x,y)$

$$\nabla g(a,b,f(a,b)) = \left[-\frac{\partial f}{\partial x}(a,b), -\frac{\partial f}{\partial y}(a,b), 1\right]$$

$\Rightarrow$ tangent plane is

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + (z - f(a,b)) = 0.$$

Note that we've already seen this plane ~~[illegible]~~ as graph of linearisation of f.