



## APRIL 30 LECTURE

### TEXTBOOK REFERENCE:

- *Vector Calculus*, Colley, 4th Edition: §4.3

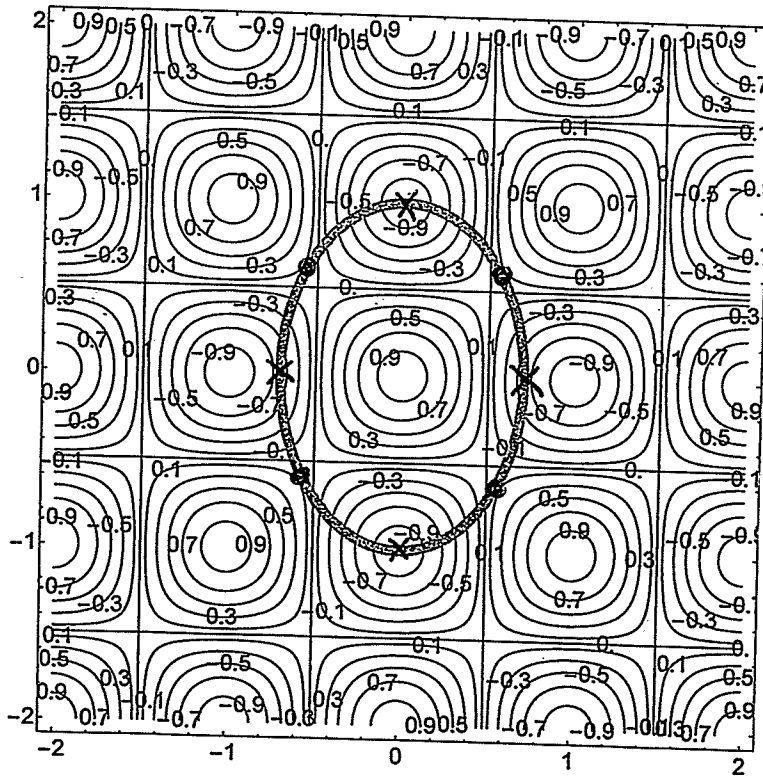
## THE METHOD OF LAGRANGE MULTIPLIERS

### LEARNING OBJECTIVES:

- Understand why the method of Lagrange multipliers works.
- Learn how to use the method of Lagrange multipliers in constrained optimisation.

KEYWORDS: the method of Lagrange multipliers

Consider the function  $f(x, y) = \cos(\pi x) \cos(\pi y)$  and the ellipse  $2x^2 + y^2 = 1$ . We plot the level curve diagram of  $f(x, y)$  below along with the ellipse.



X - LOCAL  
MIN

o - LOCAL  
MAX

### CHECK YOUR UNDERSTANDING

Imagine you are walking on the graph of  $f(x, y)$ , following the path of the ellipse.

1. Mark those points where you will encounter a local maximum/minimum on your trip.
2. What are the characteristics of these local extrema?

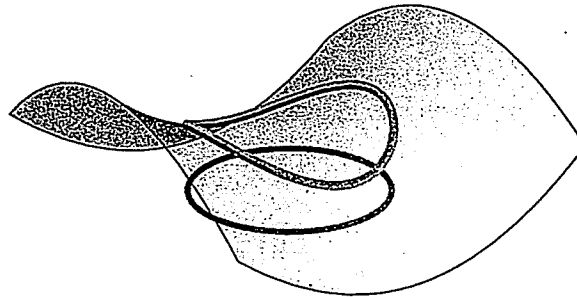
We encounter two types of constrained extrema:

**Constrained extrema:**

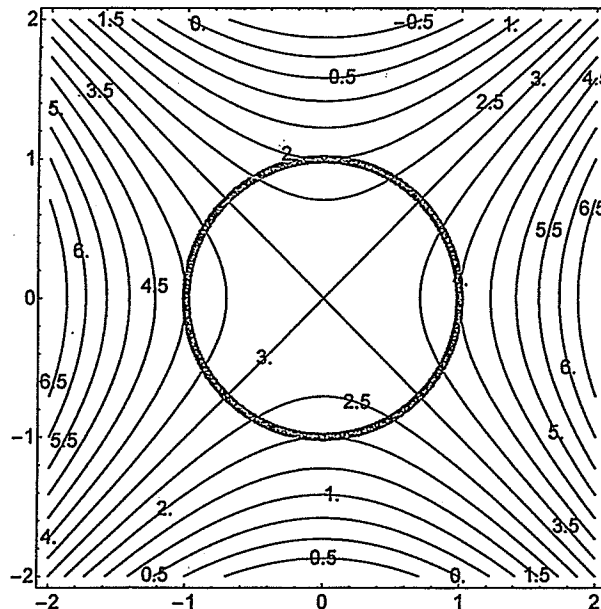
- Constrained extrema occurring as local maximum/minimum of  $f(x, y)$
- Constrained extrema occurring away from local maxima/minima of  $f(x, y)$ .

**Observation:** The additional constrained extrema occur at those  $(x, y)$  on the ellipse for which the ellipse is tangent to a level curve of  $f(x, y)$  - equivalently, the additional extrema occur at those  $(x, y)$  where  $\nabla f(x, y)$  is parallel to  $\nabla g(x, y)$ , where  $g(x, y) = 2x^2 + y^2 - 1$ .

Let's see another example: consider  $f(x, y) = x^2 - y^2 + 3$  and the unit circle  $x^2 + y^2 = 1$ . We are interested in understanding where  $f(x, y)$  admits local maxima/minima subject to the constraint  $x^2 + y^2 = 1$ .



Plotted above is the graph of  $f(x, y)$ ; we've also marked the outputs corresponding to those inputs that lie on the unit circle. Based on the graph we expect there to be four constrained extrema.



Note that none of these constrained extrema are occurring at a local maximum/minimum of  $f(x, y)$  (there is a single local extrema of  $f(x, y)$  - where? what's its nature?). We confirm our observation above:

The additional constrained extrema occur at those  $(x, y)$  on the unit circle for which the circle is tangent to a level curve of  $f(x, y)$  - equivalently, the additional extrema occur at those  $(x, y)$  where  $\nabla f(x, y)$  is parallel to  $\nabla h(x, y)$ , where  $h(x, y) = x^2 + y^2 - 1$ .

### Determination of constrained extrema:

Recall: two vectors  $\underline{u}, \underline{v}$  are parallel if there exists some nonzero scalar  $\lambda$  so that  $\underline{u} = \lambda \underline{v}$ .

We compute

$$\nabla f = [2x \quad -2y], \quad \nabla h = [2x \quad 2y]$$

and these two vectors are parallel if there exists nonzero  $\lambda$  so that

$$\nabla f = \lambda \nabla g \implies 2x = \lambda 2x, \quad 2y = -\lambda 2y \quad (*)$$

If  $x \neq 0$  then

$$2x = \lambda 2x \implies \lambda = 1 \implies y = 0$$

So, for those  $(x, y)$  on the unit circle for which  $y = 0$  - i.e.  $(\pm 1, 0)$  - we have  $\nabla f$  is parallel to  $\nabla h$ .

If  $x = 0$  then the first equation in (\*) always holds, and the second equation holds for all  $y$  if  $\lambda = -1$ . Hence, for those  $(x, y)$  on the unit circle for which  $x = 0$  - i.e.  $(0, \pm 1)$  - we have  $\nabla f$  is parallel to  $\nabla h$ .

Hence, we have four constrained extrema:  $(\pm 1, 0), (0, \pm 1)$ . To determine if we have a maximum/minimum we input these points into  $f(x, y)$ : since

$$f(\pm 1, 0) = 4, \quad f(0, \pm 1) = 2$$

we have  $(\pm 1, 0)$  are constrained maxima and  $(0, \pm 1)$  are constrained minima.

### MATHEMATICAL WORKOUT

Follow the method above to determine the constrained extrema of  $f(x, y) = 4x - 3y$  subject to the constraint  $x^2 + y^2 = 25$ . Can you think of another approach to determine the constrained extrema?

$$\nabla f = [4 \quad -3]$$

$$\nabla g = [2x \quad 2y]$$

Find  $(x, y)$  ~~and~~ <sup>and</sup>  $\lambda \neq 0$  s.t.

$$\textcircled{1} \quad x^2 + y^2 = 25$$

$$\textcircled{2} \quad 2\lambda x = 4 \implies \lambda x = 2$$

$$\textcircled{3} \quad 2\lambda y = -3 \implies \lambda y = -\frac{3}{2}$$

Let  $g(x, y) = x^2 + y^2 - 25$

•  $x \neq 0$ : ①  $\Rightarrow \lambda = \frac{2}{x}$

Sub. into ②  $\Rightarrow \frac{2}{x} \cdot y = -\frac{3}{2} \Rightarrow y = -\frac{3}{4}x$

Sub. into ③  $\Rightarrow x^2 + \left(\frac{-3}{4}x\right)^2 = 25$

$\Rightarrow$   ~~$x^2$~~   $x^2 + \frac{9}{16}x^2 = 25$

$\Rightarrow \frac{25}{16}x^2 = 25$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

$\Rightarrow y = \mp \frac{3}{4} \cdot 4 = \mp 3$

Hence,  $(4, -3)$ ,  $(-4, 3)$  are constrained extrema.

•  $x = 0$ : No sub solution to ①:  $0 = 4$

Hence, can't have  $x = 0$ .

$\Rightarrow$  ~~the~~ constrained maxima are  $(4, -3)$ ,  $(-4, 3)$