Middlebury College

Multivariable Calculus
Spring 2018
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# April 30 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §4.3

## The Method of Lagrange Multipliers

Learning Objectives:
- *Understand why the method of Lagrange multipliers works.*
- *Learn how to use the method of Lagrange multipliers in constrained optimisation.*

Keywords: the method of Lagrange multipliers

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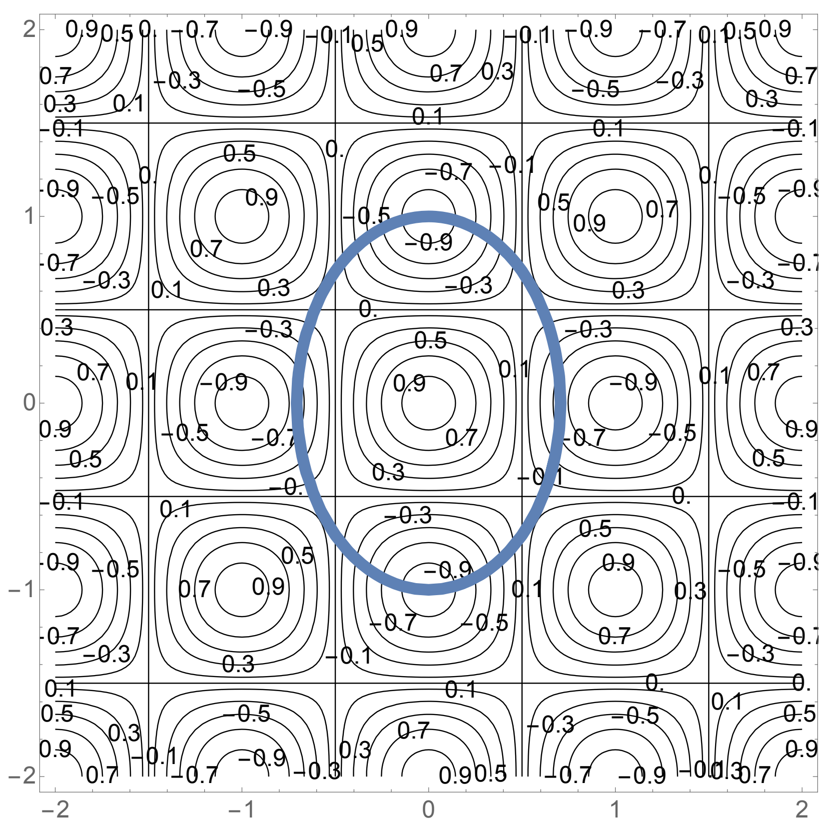
Consider the function $f(x, y) = \cos(\pi x)\cos(\pi y)$ and the ellipse $2x^2 + y^2 = 1$. We plot the level surve diagram of $f(x, y)$ below along with the ellipse.




Check your understanding
Image you are walking on the graph of $f(x, y)$, following the path of the ellipse.

1. Mark those points where you will encounter a local maximum/minimum **on your trip**.

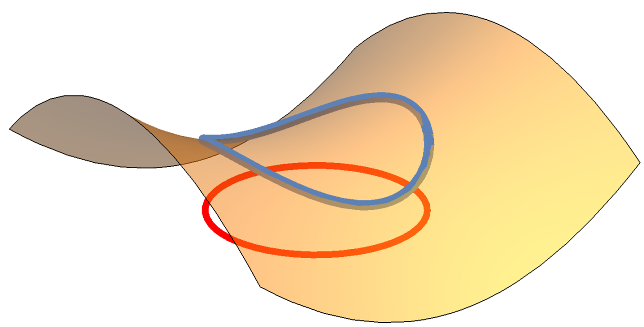
2. What are the characteristics of these local extrema?

We encounter two types of **constrained extrema**:

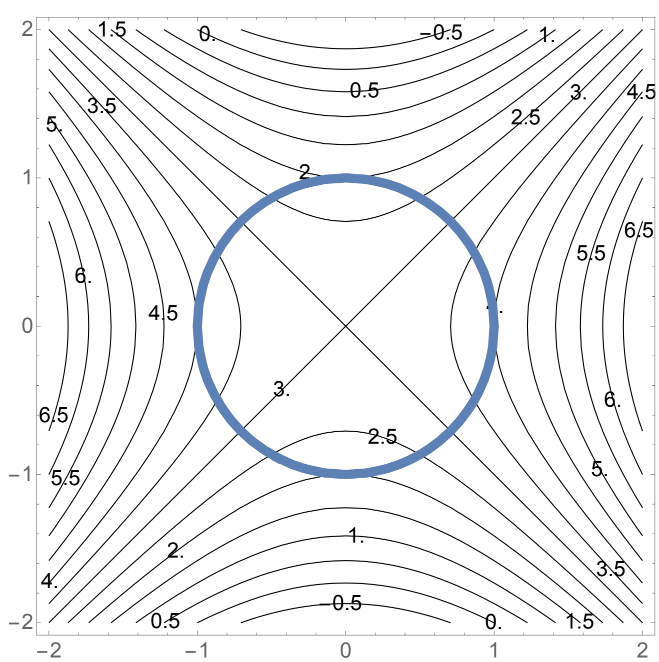
> **Constrained extrema:**
>
> - Constrained extrema occurring as local maximum/minimum of $f(x, y)$
> - Constrained extrema occurring away from local maxima/minima of $f(x, y)$.

**Observation:** The additional constrained extrema occur at those $(x, y)$ on the ellipse for which the ellipse is tangent to a level curve of $f(x, y)$ - equivalently, the additional extrema occur at those $(x, y)$ where $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$, where $g(x, y) = 2x^2 + y^2 - 1$.

Let's see another example: consider $f(x, y) = x^2 - y^2 + 3$ and the unit circle $x^2 + y^2 = 1$. We are interested in understanding where $f(x, y)$ admits local maxima/minima subject to the constraint $x^2 + y^2 = 1$.



Plotted above is the graph of $f(x, y)$; we've also marked the outputs corresponding to those inputs that lie on the unit circle. Based on the graph we expect there to be four **constrained extrema**.



Note that none of these constrained extrema are occuring at a local maximum/minimum of $f(x,y)$ (there is a single local extrema of $f(x,y)$ - where? what's its nature?). We confirm our observation above:

> The additional constrained extrema occur at those $(x, y)$ on the unit circle for which the circle is tangent to a level curve of $f(x,y)$ - equivalently, the additional extrema occur at those $(x,y)$ where $\nabla f(x,y)$ is parallel to $\nabla h(x,y)$, where $h(x,y) = x^2 + y^2 - 1$.

## Determination of constrained extrema:

**Recall:** two vectors $\underline{u}, \underline{v}$ are parallel if there exists some nonzero scalar $\lambda$ so that $\underline{u} = \lambda \underline{v}$.

We compute

$$\nabla f = \begin{bmatrix} 2x & -2y \end{bmatrix}, \qquad \nabla h = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

and these two vectors are parallel if there exists nonzero $\lambda$ so that

$$\nabla f = \lambda \nabla g \quad \Longrightarrow \quad 2x = \lambda 2x, \quad 2y = -\lambda 2y \qquad (*)$$

If $x \neq 0$ then

$$2x = \lambda 2x \quad \Longrightarrow \quad \lambda = 1 \quad \Longrightarrow \quad y = 0$$

So, for those $(x,y)$ on the unit circle for which $y = 0$ - i.e. $(\pm 1, 0)$ - we have $\nabla f$ is parallel to $\nabla h$.

If $x = 0$ then the first equation in $(*)$ always holds, and the second equation holds for all $y$ if $\lambda = -1$. Hence, for those $(x,y)$ on the unit circle for which $x = 0$ - i.e. $(0, \pm 1)$ - we have $\nabla f$ is parallel to $\nabla h$.

Hence, we have four constrained extrema: $(\pm 1, 0)$, $(0, \pm 1)$. To determine if we have a maximum/minimum we input these points into $f(x,y)$: since

$$f(\pm 1, 0) = 4, \qquad f(0, \pm 1) = 2$$

we have $(\pm 1, 0)$ are constrained maxima and $(0, \pm 1)$ are constrained minima.

Mathematical workout

Follow the method above to determine the constrained extrema of $f(x,y) = 4x - 3y$ subject to the constraint $x^2 + y^2 = 25$. Can you think of another approach to determine the constrained extrema?