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April 27 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §4.1, 4.2

The Second Derivative Test; The Hessian

Learning Objectives:
- *Learn how to use the Second Derivative Test.*
- *Learn what the Hessian matrix is.*

Keywords: Second Derivative Test, Hessian matrix

The Second Derivative Test

Let $f : X \subset \mathbb{R}^2$ be a differentiable function with continuous (mixed) second order partial derivatives, $\underline{a} = (a, b) \in X$. In the last lecture we introduced the **second order Taylor polynomial of f near $\underline{a}$**

$$p_2(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \frac{1}{2} f_{xx}(a, b)(x - a)^2$$

$$+ f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{yy}(a, b)(y - b)^2$$

The second order Taylor polynomial $p_2(x, y)$ is a good approximation of $f(x, y)$ near $\underline{a}$ in the following sense:

$$\lim_{\underline{x} \to \underline{a}} \frac{f(\underline{x}) - p_2(\underline{x})}{|\underline{x} - \underline{a}|^2} = 0 \qquad (*)$$

Suppose that $\underline{a} = (0, 0) \in X$ is a critical point, so that $\nabla f(0, 0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$ (this is not a restrictive assumption on $f(x, y)$ - we could always ensure this to be the case once we perform a translation change-of-coordinates). Then,

$$p_2(x, y) = \frac{1}{2} \left(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2 \right)$$

Write

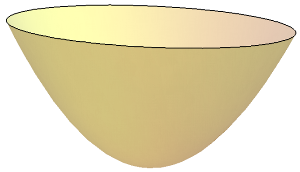
$$\alpha = \frac{\partial^2 f}{\partial x^2}(0, 0), \quad \beta = \frac{\partial^2 f}{\partial x \partial y}(0, 0), \quad \gamma = \frac{\partial^2 f}{\partial y^2}$$

so that

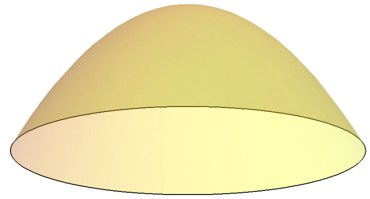
$$p_2(x, y) = \frac{1}{2} \left(\alpha x^2 + 2\beta xy + \gamma y^2 \right)$$

Assume $\alpha \neq 0$ (this need not hold, in general) and $f(0, 0) = 0$ (this may be assumed without loss of generality). Then, upon **completing the square** we find

$$\begin{aligned} p_2(x, y) &= \frac{1}{2} \left(\alpha x^2 + 2\beta xy + \gamma y^2 \right) \\ &= \alpha \left(x + \frac{\beta y}{\alpha} \right)^2 + y^2 \left(\frac{\alpha\gamma - \beta^2}{\alpha} \right) \end{aligned}$$

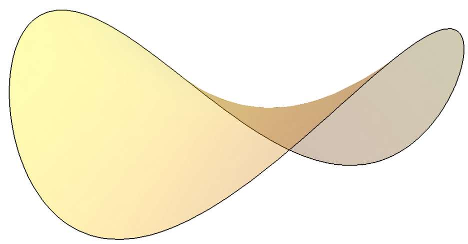


$\alpha, \delta > 0$



$\alpha, \delta < 0$

Define $\delta = \frac{\alpha\gamma - \beta^2}{\alpha}$ and **assume** $\delta \neq 0$. We have the following cases for what the graph $z = p_2(x, y)$ looks like near $(0, 0)$:



$\alpha > 0, \delta < 0 \quad$ or $\quad \alpha < 0, \delta > 0$

Remark:

1. If we make the linear change of coordinates $\overline{x} = x + \frac{\beta y}{2\alpha}$ then

$$p_2(\overline{x}, y) = \alpha\overline{x}^2 + \delta y^2$$

2. If $\alpha = 0$ then we complete the square with respect to y, obtaining:

$$p_2(x, y) = \gamma\left(y + \frac{\beta}{\gamma}x\right)^2 - \frac{\beta^2}{\gamma}y^2$$

   In particular, whenever $\alpha = 0$ and $\beta \neq 0$ the graph $z = p_2(x, y)$ near $(0, 0)$ looks like a saddle.

Now, $(*)$ implies that, for (x, y) close to $(0, 0)$,

$$f(x, y) - f(0, 0) \approx p_2(\underline{x}) - f(0, 0) = \frac{1}{2}\left(\alpha x^2 + 2\beta xy + \gamma y^2\right)$$

$$\implies f(x, y) - f(0, 0) \approx \frac{1}{2}\left(\alpha\left(x + \frac{\beta y}{\alpha}\right)^2 + \delta y^2\right)$$

Hence, for (x, y) near the critical point $(0, 0)$, we have the following characterisation of the nature of the critical point:

Derivative info.	Nature of crit. pt.
$\alpha, \delta > 0$	local min.
$\alpha, \delta < 0$	local max.
$\alpha < 0, \delta > 0$ or $\alpha > 0, \delta < 0$	saddle
$\alpha = 0,\ \beta \neq 0$	saddle

We've just exhibited the following **second derivative test** for determining the nature of a critical point of $f(x, y)$:

Second Derivative Test:

Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function with continuous second order partial derivatives. Let $\underline{a} = (a, b)$ be a critical point of f. Define

$$\alpha = \frac{\partial^2 f}{\partial x^2}(a, b) \quad \beta = \frac{\partial^2 f}{\partial x \partial y}(a, b), \quad \gamma = \frac{\partial^2 f}{\partial y^2}(a, b)$$

$$\alpha = 0,\ \beta \neq 0 \quad \longleftrightarrow \quad \text{saddle}$$

Define

$$\delta = \frac{\alpha\gamma - \beta^2}{\alpha}$$

If $\delta \neq 0$ then

$$\begin{aligned} \alpha, \delta > 0 \quad &\longleftrightarrow \quad \text{local min.} \\ \alpha, \delta < 0 \quad &\longleftrightarrow \quad \text{local max.} \\ \alpha < 0, \delta > 0 \ \textbf{or} \ \alpha > 0, \delta < 0 \quad &\longleftrightarrow \quad \text{saddle} \end{aligned}$$

If $\alpha\gamma - \beta^2 = 0$ then we say $\underline{a}$ is **degenerate** and we can't deduce the nature of the critical point using the second derivative test.

Example:

1. Consider the function $f(x, y) = x^2 + 2x + y^2$. Then,

$$\nabla f = \begin{bmatrix} 2x + 2 & 2y \end{bmatrix}$$

The critical points are those (x, y) where $\nabla f = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\implies 2(x + 1) = 0 \quad \text{and} \quad 2y = 0$$

There is a single critical point $(-1, 0)$. We compute

$$\alpha = 2, \quad \beta = 0, \quad \gamma = 2$$

Then,

$$\delta = \frac{\alpha\gamma - \beta^2}{\alpha} = 2.$$

Hence, since $\alpha, \delta > 0$ the critical point $(-1, 0)$ is a local minimum.

2. Let $f(x,y) = x^3 - 3xy^2$. Let's determine the nature of the critical points. First

$$\nabla f = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \end{bmatrix}$$

Then, critical points are those (x,y) satisfying

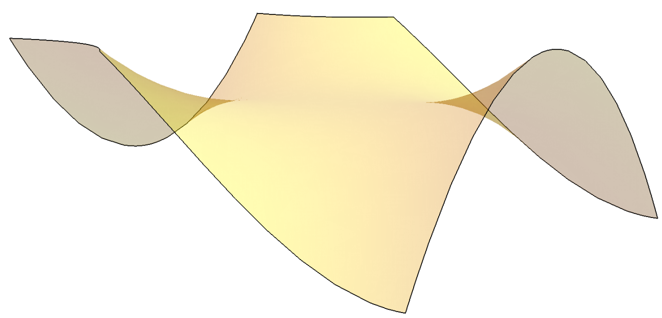
$$3x^2 - 3y^2 = 0 \qquad \text{and} \qquad 6xy = 0$$

The second equation gives $x = 0$ or $y = 0$. Substituting $x = 0$ into the first equation gives $y = 0$. Substituting $y = 0$ into the first equation gives $x = 0$. Hence, there is exactly one critical point at $(0,0)$. Then,

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0$$

Since $\alpha\gamma\beta^2 = 0$ the critical point $(0,0)$ is degenerate and we need further analysis to determine its nature.

The graph $z = x^3 - 3xy^2$ is known as the **monkey saddle**:



3. Let $f(x,y) = x^2y - 2xy^2 + 3xy + 4$. Let's determine the nature of the critical points. First, compute

$$\nabla f = \begin{bmatrix} 2xy - 2y^2 + 3y & x^2 - 4xy + 3x \end{bmatrix}$$

Thus, the critical points are those (x,y) such that

$$0 = 2xy - 2y^2 + 3y = y(2x - 2y + 3), \qquad 0 = x^2 - 4xy + 3x = x(x - 4y + 3)$$

The first equation holds when either $y = 0$ or $2x - 2y + 3 = 0$.

• $y = 0$: if $y = 0$ then the second equation becomes $0 = x(x+3)$. Hence, $(0,0)$ and $(-3,0)$ are critical points.

• $2x - 2y + 3 = 0$: Then, $x = y - 3/2$. Substitute this into the second equation to obtain

$$y - 3/2 = 0 \qquad \text{or} \qquad y - 3/2 - 4y + 3 = 0 \qquad \Longrightarrow \qquad y = 3/2 \quad \text{or} \quad y = 1/2$$

Hence, $(0,3/2)$ and $(-1,1/2)$ are critical points. We check the nature of each of these critical points:

- $(0,0)$: we compute

$$\alpha = 0, \quad \beta = 3, \quad \gamma = 0$$

Since $\alpha = 0$, $\beta \neq 0$ then $(0,0)$ is a saddle.

- $(-3,0)$: we compute

$$\alpha = 0, \quad \beta = -3, \quad \gamma = 12$$

Since $\alpha = 0$, $\beta \neq 0$ and $(-3,0)$ is a saddle.

- $(0,3/2)$: we compute

$$\alpha = 3, \quad \beta = -3, \quad \gamma = 0$$

Then, $\delta = \frac{\alpha\gamma - \beta^2}{\alpha} = -3$. Since $\alpha > 0$, $\delta < 0$ we have $(0,3/2)$ is a saddle.

- $(-1,1/2)$: we compute

$$\alpha = 1, \quad \beta = -1, \quad \gamma = -4$$

Then, $\delta = \frac{\alpha\gamma - \beta^2}{\alpha} = -5$. Since $\alpha > 0$, $\delta < 0$ we have $(-1,1/2)$ is a saddle.

The Hessian

Define the Hessian of $f(x,y)$ at $\underline{a} = (a,b)$ to be the 2×2 matrix

$$Hf(\underline{a}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{bmatrix}$$

Using the notation above, and Clairaut's Theorem,

$$Hf(\underline{a}) = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

Then, the Second Derivative Test can be restated as follows: let h_{ij} be the ij-entry of $H = Hf(\underline{a})$.

Hessian.	Nature of crit. pt.
$h_{11}, \det H > 0$	local min.
$h_{11} < 0, \det H > 0$	local max.
$\det H < 0$	saddle
$\det H = 0$	degenerate

The Hessian H appears naturally in the following setting: the second order Taylor polynomial near $\underline{a}$ can be written

$$p_2(\underline{x}) = f(\underline{x}) + \frac{1}{2}(\underline{x} - \underline{a})^t H (\underline{x} - \underline{a})$$

Remark: The Second Derivative Test can be extended to scalar-valued functions $f : X \subset \mathbb{R}^n \to \mathbb{R}$ of n variables. See p. 268 of the textbook.