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April 23 Lecture

Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §4.1, 4.2

Extrema for Several Variable Functions

Learning Objectives:
- *Learn the definition of local maximum, local minimum, critical point.*
- *Learn the statement of Taylor's Second Order Formula.*
- *Learn how to compute the second order Taylor polynomial.*

Keywords: local maximum, local minimum, critical point, second order Taylor polynomial

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Maxima & Minima

For the next few lectures we will see how (partial) derivatives of functions of several variables f can be used to determine properties of f.

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.

- We say that $\underline{a} \in X$ is a **local minimum** if there exists $r > 0$ such that
$$f(\underline{a}) \leq f(\underline{x}) \quad \text{whenever } |\underline{x} - \underline{a}| < r.$$
- We say that $\underline{a}$ is a **local maximum** if there exists $r > 0$ such that
$$f(\underline{x}) \leq f(\underline{a}) \quad \text{whenever } |\underline{x} - \underline{a}| < r.$$

Example: Let $f(x,y) = x^2 + y^2$. Then, $(0,0)$ is a local minimum: take $r = 1$ (for example) then, for any $\underline{x} = (x,y)$ such that $|\underline{x}| = \sqrt{x^2+y^2} < 1$ we have $f(x,y) = x^2 + y^2 \geq 0 = f(0,0)$.

Remark: As we will soon see, this example is indicative of the behaviour of a function near a local minimum.

Suppose that $\underline{a} = (a,b) \in \mathbb{R}^2$ is a local maximum of the **differentiable** function $f(x,y)$. Then, for $h \in \mathbb{R}$ sufficiently small, we have
$$f(a+h,b) - f(a,b) \leq 0.$$
In particular, if $h > 0$ is sufficiently small then
$$\frac{f(a+h,b) - f(a,b)}{h} \leq 0 \quad \Longrightarrow \quad \frac{\partial f}{\partial x}(a,b) \leq 0$$
Meanwhile, if $h < 0$ is sufficiently small then
$$\frac{f(a+h,b) - f(a,b)}{h} \geq 0 \quad \Longrightarrow \quad \frac{\partial f}{\partial x}(a,b) \geq 0$$
Hence, $\frac{\partial f}{\partial x}(a,b) = 0$. A similar argument shows that $\frac{\partial f}{\partial y}(a,b) = 0$.

Remark:

1. Proceeding as above, we can show that if $\underline{a}$ is a local minimum of f then $\frac{\partial f}{\partial x}(\underline{a}) = \frac{\partial f}{\partial y}(\underline{a}) = 0$.

2. An analogous result holds more generally:

> **Multivariable Fermat's Theorem**
>
> Suppose that $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable. If $\underline{a}$ is a local maximum/minimum of f then $\nabla f(\underline{a}) = \underline{0}$.

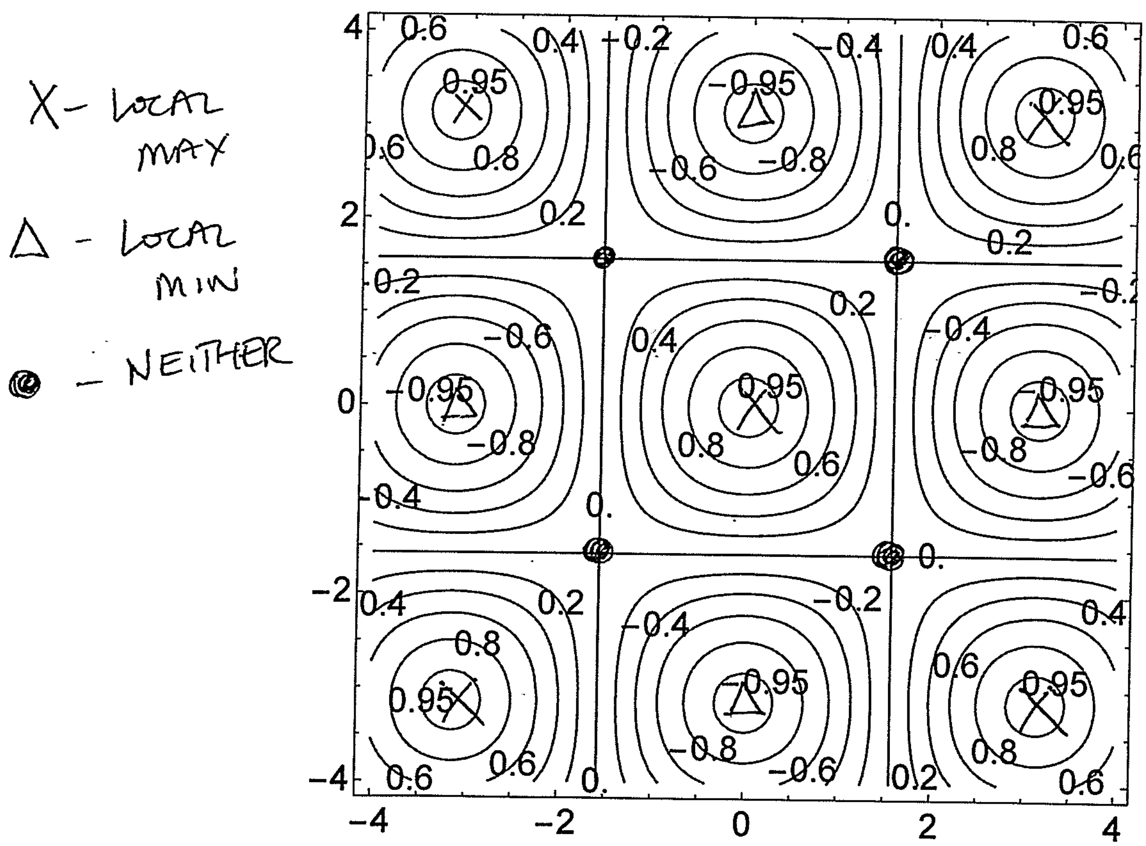
A point $\underline{a} \in X$ such that $\nabla f(\underline{a}) = \underline{0}$, or f is not differentiable at $\underline{a}$ is called a **critical point of** f.

We have an approach to determining local maxima/minima of a function f:

- Determine the critical points of f;
- Check whether the critical points just found are local maxima/minima.

Consider the following level curve diagram of

$$f(x, y) = \cos(x)\cos(y), \qquad -4 \leq x, y \leq 4$$



Exercise: Using the level curve diagram

- indicate all the local maxima/minima (there are nine points altogether);
- find a critical point that is not a local maximum/minimum (there are four critical points that are not local max/min).

How would we determine these points without the level curve diagram? First we determine the critical points of $f(x,y) = \cos(x)\cos(y)$ - these are those points (a,b) in the domain of f satisfying

$$\begin{bmatrix} 0 & 0 \end{bmatrix} = \nabla f(a,b) = \begin{bmatrix} -\sin(a)\cos(b) & -\cos(a)\sin(b) \end{bmatrix}$$
$$\implies \quad \sin(a)\cos(b) = 0 = \cos(a)\sin(b)$$
$$\implies (a,b) = (r\pi/2, s\pi/2), \quad r,s \in \{-2,-1,0,1,2\}$$

(*Recall that the domain of f is $\{(x,y) \mid -4 \leq x,y \leq 4\}$*)
For each of these critical points we check if they are local maximum/minimums. For example, if $(a,b) = (0,0)$ let's check how $f(x,y)$ changes as we move away from $(0,0)$: for h,k sufficiently small and nonzero

$$f(h,k) - f(0,0) = \cos(h)\cos(k) - \cos(0)\cos(0) = \cos(h)\cos(k) - 1 < 0$$

Hence, $(0,0)$ is a **local maximum**. Using the double angle formula $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$, a similar computation shows that

$$\begin{array}{ll} \text{local maximum:} & (0,0), (-\pi,-\pi), (-\pi,\pi), (\pi,-\pi), (\pi,\pi) \\ \text{local minimum:} & (\pm\pi, 0), (0, \pm\pi) \end{array}$$

What about the critical point $(\pi/2, \pi/2)$? This point is neither a local maximum nor a local minimum. The level curve diagram indicates this point is not a local maximum or local minimum. Let's check how $f(x,y)$ changes as we move away from $(\pi/2,\pi/2)$: for h,k sufficiently small and nonzero

$$f(\pi/2+h, \pi/2+k) - f(\pi/2,\pi/2) = \cos(\pi/2+h)\cos(\pi/2+k) - \cos(\pi/2)\cos(\pi/2)$$
$$= \sin(h)\sin(k)$$

In particular:

$$f(\pi/2+h, \pi/2+k) - f(\pi/2,\pi/2) \begin{cases} < 0, & \text{if } h > 0, k < 0 \text{ or } h < 0, k > 0 \\ > 0, & \text{if } h,k > 0 \text{ or } h,k < 0 \end{cases}$$

Near to the critical point $(\pi/2,\pi/2)$, $f(x,y)$ is both strictly increasing and strictly decreasing.

> **Question:**
>
> Is there a methodical way to determine whether a critical point is a local maximum/minimum/neither of a differentiable function f?

Taylor's Theorem: Second Order Formula

Let $f(x,y)$ be a function with continuous second partial (mixed) derivatives having domain $X \subset \mathbb{R}^2$. We've already seen a first order approximation to $f(x,y)$ near to $(a,b) \in X$: this is the **linearisation $L(x,y)$ of $f(x,y)$ near to (a,b)**

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

If (a,b) is a critical point of $f(x,y)$, so that $\nabla f(a,b) = \underline{0}^t$, then $L(x,y) = f(a,b)$ is a constant function. A natural question is the following:

Question:

Is there a degree two polynomial function

$$p_2(x,y) = \alpha + \beta(x-a) + \gamma(y-b) + \delta(x-a)^2 + \epsilon(x-a)(y-b) + \eta(y-b)^2$$

that 'closely approximates' $f(x,y)$ near to (a,b)?

Suppose that such a polynomial function existed. Then, it would seem reasonable to expect

$$p_2(a,b) = \alpha$$

$$\frac{\partial p_2}{\partial x}(a,b) = \beta, \quad \frac{\partial f}{\partial y}(a,b) = \gamma$$

$$\frac{\partial^2 f}{\partial x^2}(a,b) = 2\delta, \quad \frac{\partial^2 f}{\partial x \partial y}(a,b) = 2\epsilon, \quad \frac{\partial^2 f}{\partial y^2}(a,b) = 2\eta$$

In fact, it's possible to show that such a polynomial does exist, in general:

Taylor's Theorem: Second Order Formula

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives, $\underline{a} = (a_1, \ldots, a_n) \in X$. Then, there exists a degree two polynomial $p_2(\underline{x})$, called the **second order Taylor polynomial of** $f(x,y)$ **near** $\underline{a}$, such that

$$\lim_{\underline{x}\to\underline{a}} \frac{|f(\underline{x}) - p_2(\underline{x})|}{|\underline{x} - \underline{a}|^2} = 0$$

Moreover,

$$p_2(\underline{x}) = f(\underline{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\underline{a})(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{a})(x_i - a_i)(x_j - a_j) \quad (*)$$

Remark:

1. For a function $f(x,y)$ of two variables

$$p_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2$$
$$+ f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$$

2. $(*)$ can be written in the compact form

$$p_2(\underline{x}) = f(\underline{a}) + \nabla f(\underline{a})(\underline{x} - \underline{a}) + \frac{1}{2}(\underline{x} - \underline{a})^t Hf(\underline{a})(\underline{x} - \underline{a})$$

where $Hf(\underline{a})$ is an $n \times n$ symmetric matrix called the **Hessian of** f. We will introduce the Hessian in the next lecture. For $n = 2$, the Hessian is

$$Hf(a,b) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial y \partial x}(a,b) \\ \frac{\partial^2 f}{\partial x \partial y}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{bmatrix}$$

Example:

1. Consider $f(x, y) = \cos(x)\cos(y)$, $(a, b) = (0, 0)$. Then,

   $f(0,0) = 1, \quad f_x(0,0) = f_y(0,0) = 0, \quad f_{xx}(0,0) = f_{yy}(0,0) = -1, \quad f_{xy}(0,0) = 0$

   Hence, the second order Taylor polynomial of $f(x, y)$ near $(0, 0)$

   $$p_2(x) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

   ← $P_2(x) - f(0,0) = -\frac{1}{2}x^2 - \frac{1}{2}y^2$ ⇒ near to $(0,0)$ $f(x,y)$ "looks like" upside-down paraboloid

2. Consider the function $f(x, y) = x^3 + 3xy + y^3$. Then,

   $$f(1,1) = 5, \quad \frac{\partial f}{\partial x}(1,1) = 6 = \frac{\partial f}{\partial y}(1,1),$$

   $$\frac{\partial^2 f}{\partial x^2}(1,1) = 6 = \frac{\partial^2 f}{\partial y^2}(1,1), \quad \frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{\partial^2 f}{\partial y \partial x}(1,1) = 3$$

   Hence, the second order Taylor polynomial near to $(1, 1)$ is

   $$p_2(x, y) = 5 + 6(x-1) + 6(y-1) + 3(x-1)^2 + 3(x-1)(y-1) + 3(y-1)^2$$

Next time: We will see how to use the second order Taylor polynomial to determine the nature of a critical point.