Middlebury College

Multivariable Calculus
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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §4.2

The Solution to the Potential Function Problem

Learning Objectives:
- *Understand the Solution to the Potential Function Problem.*
- *Learn the definition of local maximum, local minimum, critical point.*

Keywords: Solution to Potential Function Problem, local maximum, local minimum, critical point

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Summary of Potential Function Problem

Let $\underline{F} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ be a continuous vector field on $X \subseteq \mathbb{R}^2$.

Potential Function Problem

Under what conditions does there exist a potential function f for $\underline{F}$ i.e. so that $\nabla f = \underline{F}$?

We saw a (local) solution to the Potential Function Problem:

Local Solution to Potential Function Problem

Suppose X is $\mathbb{R}^2$ or an open disc/rectangle. If $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ then there exists a potential function f for $\underline{F}$.

Example: The following non-conservative vector field highlights the importance of the assumption on the domain:

$$\underline{F} = \begin{bmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$$

When X is a more general subset of $\mathbb{R}^2$ we've seen the following necessary conditions:

Necessary Conditions for Potential Functions

Suppose $\nabla f = \underline{F}$. Then,

1. $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.
2. $\int_C \underline{F} \cdot d\underline{s} = \int_{C'} \underline{F} \cdot d\underline{s}$, for any two (oriented) curves in X with same start/end point (**independence of path property**)
3. $\int_C \underline{F} \cdot d\underline{s} = 0$, for any (oriented) closed curve in X (**closed curve property**)

Remark: the vector field $\underline{F}$ given above fails property 3.: the vector line integral of $\underline{F}$ along the unit circle is 2π.

It turns out that conditions 2. and 3. are equivalent:

- Suppose that 2. holds. Let C be an (oriented) closed curve in X. Choose $P, Q \in C$, $P \neq Q$. Define C_1 to be the the (oriented) segment of C starting at P and ending at Q, and let C_2 be the (oriented) segment of C starting at Q and ending at P. Write $(C_2)_{opp}$ for C_2 but with the reversed orientation, so that $(C_2)_{opp}$ starts at P and ends at Q. In particular, both C_1 and $(C_2)_{opp}$ are (oriented) curves in X starting/ending at the same points.

  Then, we can think of C as the piecewise curve (C_1, C_2). Hence,

  $$\int_C \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} + \int_{C_2} \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} - \int_{(C_2)_{opp}} \underline{F} \cdot d\underline{s} = 0$$

  by the independence of path property.

- Suppose that 3. holds. Let C_1 and C_2 be two (oriented) curves in X both starting at P and ending at Q. Define the piecewise (oriented) closed curve $C = (C_1, (C_2)_{opp})$. Then, using the closed curve property we have

  $$0 = \int_C \underline{F} \cdot d\underline{s} = \int_{C_1} \underline{F} \cdot d\underline{s} + \int_{(C_2)_{opp}} \underline{F} \cdot d\underline{s} \quad \Longrightarrow \quad \int_{C_1} \underline{F} \cdot d\underline{s} = \int_{C_2} \underline{F} \cdot d\underline{s}$$

Remarkably, the equivalent **independence of path** and **closed curve** properties completely characterise conservative vector fields.

Solution to Potential Function Problem

Suppose that X, the domain of $\underline{F}$, has the following property: *any two points $P, Q \in X$ can be joined by a C^1-path (equiv. an oriented curve).*
The following conditions are equivalent:

1. There exists a potential function f for $\underline{F}$.
2. $\underline{F}$ has the independence of path property.
3. $\underline{F}$ has the closed curve property.

Sketch of Proof: It suffices to show that 2. $\Longrightarrow$ 1..

Pick an arbitrary point $\underline{a} \in X$. Define, for any $\underline{x} \in X$,

$$f(\underline{x}) = \int_C \underline{F} \cdot d\underline{s}$$

where C is an (oriented) curve in X joining $\underline{a}$ to $\underline{x}$. By the independence of path property f is well-defined (i.e. it does not depend on the curve C) so we may write (taking some liberty with notation)

$$f(\underline{x}) = \int_{\underline{a}}^{\underline{x}} \underline{F} \cdot d\underline{s}$$

We need to show that f is differentiable and $\frac{\partial f}{\partial x} = u$, $\frac{\partial f}{\partial y} = v$.
Let $e_1 = (1,0) \in \mathbb{R}^2$. By definition

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_{\underline{a}}^{\underline{x}+he_1} \underline{F}\cdot d\underline{s} - \int_{\underline{a}}^{\underline{x}} \underline{F}\cdot d\underline{s}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_{\underline{x}}^{\underline{x}+he_1} \underline{F}\cdot d\underline{s} + \int_{\underline{a}}^{\underline{x}} \underline{F}\cdot d\underline{s} - \int_{\underline{a}}^{\underline{x}} \underline{F}\cdot d\underline{s}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\int_{\underline{x}}^{\underline{x}+he_1} \underline{F}\cdot d\underline{s}\end{aligned}$$

By independence of path property of $\underline{F}$ we can determine this vector line integral along any path from $\underline{x}$ to $\underline{x}+he_1$. For h sufficiently small we choose the straight line segment path $\underline{x}+te_i$, $0 \le t \le h$. Then,

$$\underline{F}(\underline{x}+te_i)\cdot e_1 = u(\underline{x}+te_i)$$

and

$$\lim_{h\to 0} \frac{1}{h}\int_{\underline{x}}^{\underline{x}+he_1} \underline{F}\cdot d\underline{s} = \lim_{h\to 0} \frac{1}{h}\int_0^h u(\underline{x}+te_1)dt$$

Now, an application of Fundamental Theorem of Calculus (here's where we require that $\underline{F}$ is continuous) gives

$$\lim_{h\to 0} \frac{1}{h}\int_0^h u(\underline{x}+te_1)dt = u(\underline{x})$$

That is, $\frac{\partial f}{\partial x} = u$. A similar argument shows $\frac{\partial f}{\partial y} = v$.

Remark: The Solution to the Potential Function Problem given above holds more generally for continuous vector fields with domain $X \subseteq \mathbb{R}^n$.

Maxima & Minima

For the next few lectures we will see how (partial) derivatives of functions of several variables f can be used to determine properties of f.

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.

- We say that $\underline{a} \in X$ is a **local minimum** if there exists $r > 0$ such that

$$f(\underline{a}) \le f(\underline{x}) \quad \text{whenever } |\underline{x}-\underline{a}| < r.$$

- We say that $\underline{a}$ is a **local maximum** if there exists $r > 0$ such that

$$f(\underline{x}) \le f(\underline{a}) \quad \text{whenever } |\underline{x}-\underline{a}| < r.$$

Example: Let $f(x, y) = x^2 + y^2$. Then, $(0, 0)$ is a local minimum: take $r = 1$ (for example) then, for any $\underline{x} = (x, y)$ such that $|\underline{x}| = \sqrt{x^2 + y^2} < 1$ we have $f(x, y) = x^2 + y^2 \geq 0 = f(0, 0)$.

Remark: As we will soon see, this example is indicative of the behaviour of a function near a local minimum.

Suppose that $\underline{a} = (a, b) \in \mathbb{R}^2$ is a local maximum of the **differentiable** function $f(x, y)$. Then, for $h \in \mathbb{R}$ sufficiently small, we have

$$f(a + h, b) - f(a, b) \leq 0.$$

In particular, if $h > 0$ is sufficiently small then

$$\frac{f(a + h, b) - f(a, b)}{h} \leq 0 \quad \Longrightarrow \quad \frac{\partial f}{\partial x}(a, b) \leq 0$$

Meanwhile, if $h < 0$ is sufficiently small then

$$\frac{f(a + h, b) - f(a, b)}{h} \geq 0 \quad \Longrightarrow \quad \frac{\partial f}{\partial x}(a, b) \geq 0$$

Hence, $\frac{\partial f}{\partial x}(a, b) = 0$. A similar argument shows that $\frac{\partial f}{\partial y}(a, b) = 0$.

Remark:

1. Proceeding as above, we can show that if $\underline{a}$ is a local minimum of f then $\frac{\partial f}{\partial x}(\underline{a}) = \frac{\partial f}{\partial y}(\underline{a}) = 0$.
2. An analogous result holds more generally:

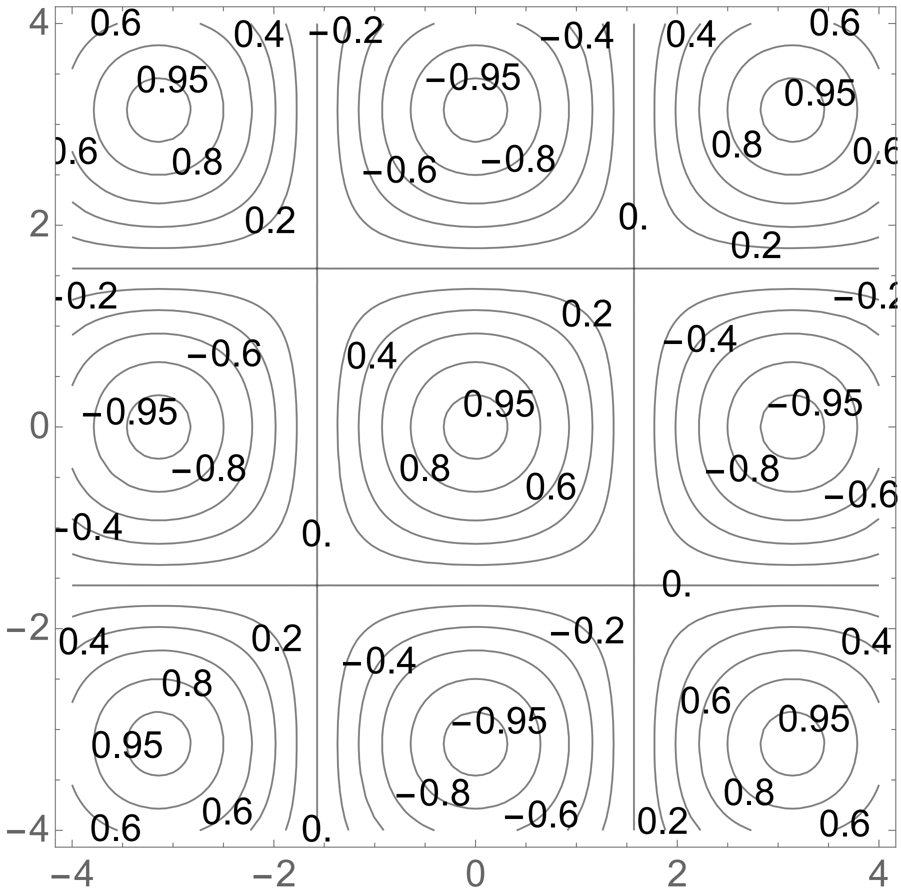
> **Multivariable Fermat's Theorem**
>
> Suppose that $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable. If $\underline{a}$ is a local maximum/minimum of f then $\nabla f(\underline{a}) = \underline{0}$.

A point $\underline{a} \in X$ such that $\nabla f(\underline{a}) = \underline{0}$, or f is not differentiable at $\underline{a}$ is called a **critical point of** f.

We have an approach to determining local maxima/minima of a function f:

- Determine the critical points of f;
- Check whether the critical points just found are local maxima/minima.

We'll see how to put this approach in practice in the next lecture. Let's investigate what to expect using the level curve diagram of $f(x, y) = \cos(x)\cos(y)$:



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