Middlebury College

Multivariable Calculus
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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §6.1

Reparameterisation

Learning Objectives:
- *Understand the notion of reparameterisation of a path.*
- *Understand the effect of reparameterisation on vector line integrals.*

Keywords: reparameterisation, vector line integrals along curves

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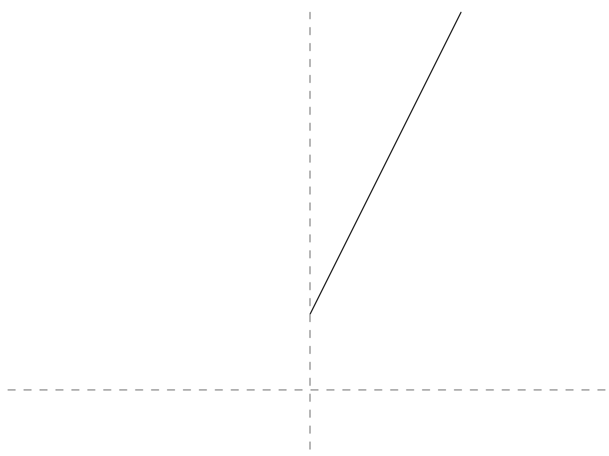
Today we will investigate the effect that **reparameterisation** has on vector line integrals.

Reparameterisations

Consider the C^1-path

$$\underline{x}(t) = \begin{bmatrix} t \\ 2t+1 \end{bmatrix}, \quad t \in [0,2]$$

whose image curve is the straight line segment between $(0,1)$ and $(2,5)$.



The same line segment may also be parameterised by the path

$$\underline{y}(t) = \begin{bmatrix} 2t \\ 4t+1 \end{bmatrix}, \quad t \in [0,1]$$

Remark: It's important to remember that $\underline{x}$ and $\underline{y}$ are **different** paths describing the same curve (i.e. the line segment).

The paths $\underline{x}(t)$ and $\underline{y}(t)$ are, of course, related:

$$\underline{x}(2t) = \underline{y}(t), \quad \underline{x}(t) = \underline{y}(t/2)$$

We say that $\underline{y}$ is a **reparameterisation of** $\underline{x}$ (and $\underline{x}$ **is reparameterisation of** $\underline{y}$). More generally:

Reparameterisation of paths

Let $\underline{x} : [a, b] \to \mathbb{R}^n$ be a C^1-path. We say that $\underline{y} : [c, d] \to \mathbb{R}^n$ is a **reparameterisation of** $\underline{x}$ if there exists a **bijective** C^1-function[1] $u : [c, d] \to [a, b]$ so that

$$\underline{y}(t) = \underline{x}(u(t)), \quad t \in [c, d]$$

Example:

1. Consider the path

$$\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad t \in [-\pi/2, \pi/2]$$

Define

$$\underline{y}(t) = \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}, \quad t \in [-\pi/8, \pi/8]$$

Then $\underline{y}$ is a reparameterisation of $\underline{x}$: if we define

$$u : [-\pi/8, \pi/8] \to [-\pi/2, \pi/2]\ ,\ t \mapsto 4t$$

then $\underline{y}(t) = \underline{x}(u(t))$. Here $u^{-1} : [-\pi/2, \pi/2] \to [-\pi/8, \pi/8], u^{-1}(s) = s/4$ is the inverse function of u.

2. The path

$$\underline{z}(t) = \begin{bmatrix} \sqrt{1-t^2} \\ t \end{bmatrix}, \quad t \in [-1, 1]$$

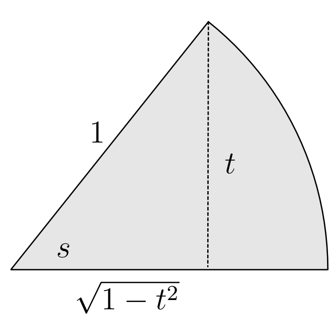
is a reparameterisation of $\underline{x}(t)$: if we define

$$u : [-1, 1] \to [-\pi/2, \pi/2]\ ,\ t \mapsto \arcsin(t)$$

then

$$\underline{x}(u(t)) = \begin{bmatrix} \cos(\arcsin(t)) \\ \sin(\arcsin(t)) \end{bmatrix} = \begin{bmatrix} \sqrt{1-t^2} \\ t \end{bmatrix} = \underline{z}(t)$$

Here we use that if $s = \arcsin(t)$ then $\sin(s) = t$ and we have the triangle



3. Let $\underline{x} : [a,b] \to \mathbb{R}^n$ be a C^1-path. Define the **opposite path** $\underline{x}_{opp} : [a,b] \to \mathbb{R}^n$ to be

$$\underline{x}_{opp}(t) \overset{def}{=} \underline{x}(a+b-t)$$

$\underline{x}_{opp}(t)$ is a reparameterisation of $\underline{x}$: we have

$$u : [a,b] \to [a,b]\ ,\ t \mapsto a+b-t$$

Observe that $\underline{x}_{opp}(a) = \underline{x}(b)$ and $\underline{x}_{opp}(b) = \underline{x}(a)$. The path $\underline{x}_{opp}(t)$ parameterises the same curve as $\underline{x}(t)$ but with **opposite direction**.

Important observations:

- If $\underline{y}$ is a reparameterisation of $\underline{x}$ then $\underline{x}$ is a reparamerisation of $\underline{y}$;
- If $\underline{y}$ is a reparameterisation of $\underline{x}$ then $\underline{y}$ and $\underline{x}$ are parameterisations of the same curve.

Suppose that $\underline{y} : [c,d] \to \mathbb{R}^n$ is a reparameterisation of $\underline{x} : [a,b] \to \mathbb{R}^n$, so that

$$\underline{y}(t) = \underline{x}(u(t)).$$

We say that $\underline{y}$ is

- **orientation-preserving** if $u(c) = a$ and $u(d) = b$,
- **orientation-reversing** if $u(c) = b$ and $u(d) = a$.

We now investigate the effects of reparameterisation on vector line integrals.

Let $\underline{F}$ be a continuous vector field on $X \subset \mathbb{R}^n$. Suppose $\underline{y}$ is a reparameterisation of $\underline{x}$ (with same notation as above), and their (common) image curve is contained in X. Then,

$$\begin{aligned}
\int_{\underline{y}} \underline{F} \cdot d\underline{s} &= \int_c^d \underline{F}(\underline{y}(t)) \cdot \underline{y}'(t) dt \\
&= \int_c^d \underline{F}(\underline{x}(u(t))) \cdot (\underline{x}'(u(t))u'(t))\, dt, \qquad \text{because } \underline{y} = \underline{x} \circ u
\end{aligned}$$

$$= \begin{cases} \int_a^b \underline{F}(\underline{x}(u) \cdot \underline{x}'(u) du, & \text{if } \underline{y} \text{ is orientation preserving} \\ \int_b^a \underline{F}(\underline{x}(u) \cdot \underline{x}'(u) du, & \text{if } \underline{y} \text{ is orientation reversing} \end{cases}$$

$$= \begin{cases} \int_{\underline{x}} \underline{F} \cdot d\underline{s}, & \text{if } \underline{y} \text{ is orientation-preserving} \\ -\int_{\underline{x}} \underline{F} \cdot d\underline{s}, & \text{if } \underline{y} \text{ is orientation-reversing} \end{cases}$$

In words,

• Vector line integrals are independent of orientation-preserving reparameterisations.
• Vector line integrals are independent (up to a sign) of orientation-reserving reparameterisations.

This allows us to **define vector line integrals of vector fields $\underline{F}$ along oriented curves** C

$$\int_C \underline{F} \cdot d\underline{s}$$

rather than along paths.

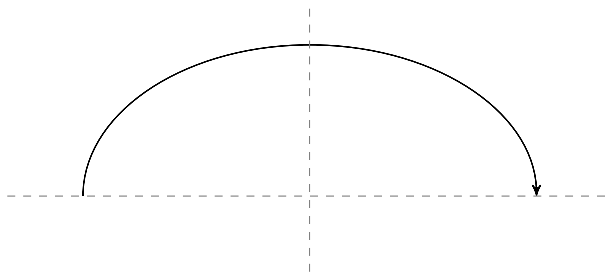
Notation: It is common to denote the vector line integral of $\underline{F} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ along C

$$\int_C u(x,y)dx + v(x,y)dy$$

Example: Consider the oriented curve C defined as the portion of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

lying in the $y \geq 0$ half-plane oriented clockwise.



We can parameterise the oriented curve C as

$$\underline{x}(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) \end{bmatrix}, \quad t \in [0, \pi]$$

The importance of what we have shown above is that, if $\underline{F}$ is a (continuous) vector field on $\mathbb{R}^2$ then

$$\int_C \underline{F} \cdot d\underline{s} \stackrel{def}{=} \int_{\underline{x}} \underline{F} \cdot d\underline{s}$$

is independent of how we parameterise the oriented curve C: we could have parameterised C by the path

$$\underline{y}(t) = \begin{bmatrix} t \\ \sqrt{4 - 4t^2/9} \end{bmatrix}, \quad t \in [-3, 3]$$

to compute the vector line integral of $\underline{F}$ along C.

Exercise: Compute

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s}$$

where $\underline{F} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$. Now, try to compute the same vector line integral of $\underline{F}$ along C using the parameterisation $\underline{y}$:

$$\int_{\underline{y}} \underline{F} \cdot d\underline{s}$$

(*This last line integral might be a bit challenging!*)