Middlebury College

Multivariable Calculus
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Textbook Reference:
- *Vector Calculus*, Colley, 4th Edition: §6.1

Line Integrals

Learning Objectives:
- *Understand the definition of a vector line integral.*
- *Learn how to compute basic vector line integrals.*
- *Learn the 'independence of path' property for line integrals of conservative vector fields*
- *Learn the Line Integral Test for non-Conservative Vector Fields*

Keywords: vector line integral, closed paths, independence of paths, Line Integral Test for non-Conservative Vector Fields

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In the previous lecture we introduced the concept of a **scalar line integral**: for a continuous function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and a C^1-path $\underline{x}(t)$, $t \in [a, b]$, we defined the **scalar line integral from a to b**

$$\int_{\underline{x}} f ds \stackrel{def}{=} \int_a^b f(\underline{x}(t))|\underline{x}(t)|dt$$

Remark:

1. Recall that a C^1-path is a differentiable path $\underline{x}(t)$ such that $\underline{x}'(t)$ is continuous.

2. We may also define scalar line integrals for a **piecewise C^1-path**: this is a collection of C^1-paths on $\mathbb{R}^n$, $(\underline{x}_1(t), \dots, \underline{x}_k(t))$, with $\underline{x}_i(t)$ defined on domain $[a_i, b_i]$, so that $\underline{x}_i(t) \in X$, for all t, and satisfying

$$\underline{x}_i(b_i) = \underline{x}_{i+1}(a_{i+1}), \quad i = 1, \dots, k-1$$

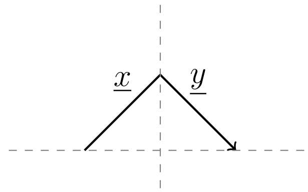
That is, the endpoint of $\underline{x}_i$ is the startpoint of $\underline{x}_{i+1}$. Then,

$$\int_{(\underline{x}_1,\dots,\underline{x}_k)} f ds \stackrel{def}{=} \sum_{i=1}^{k} \int_{\underline{x}_i} f ds$$

For example, consider the piecewise C^1-path $(\underline{x}, \underline{y})$ on $\mathbb{R}^2$, where

$$\underline{x}(t) = \begin{bmatrix} -1+t \\ t \end{bmatrix}, \quad \underline{y}(t) = \begin{bmatrix} t \\ 1-t \end{bmatrix}$$

both paths having domain $[0, 1]$.



Then, if $f(x,y) = x + y$ we have

$$\begin{aligned}
\int_{(\underline{x},\underline{y})} f ds &= \int_{\underline{x}} f ds + \int_{\underline{y}} f ds \\
&= \int_{t=0}^{t=1} f(\underline{x}(t))|\underline{x}'(t)|dt + \int_{t=0}^{t=1} f(\underline{y}(t))|\underline{y}'(t)|dt \\
&= \int_0^1 (-1+2t)\sqrt{2}dt + \int_0^1 (1-2t)\sqrt{2}dt \\
&= \sqrt{2}\int_0^1 (-1+2t+1-2t)dt = 0
\end{aligned}$$

Vector line integrals

Now we see how to integrate a vector field along a path. Let $\underline{F}$ be a continuous vector field on $\mathbb{R}^n$, $\underline{x}(t)$ a C^1-path in $\mathbb{R}^n$.

Vector line integrals

The **vector line integral of $\underline{F}$ along $\underline{x}$** is

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s} = \int_a^b \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t)dt$$

Example:

1. Let $\underline{F} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and consider the path $\underline{x}(t) = \begin{bmatrix} t \\ 2t^2 \\ t^3 \end{bmatrix}$, $t \in [0,1]$. We compute the vector line integral of $\underline{F}$ along $\underline{x}$ as follows:

$$\begin{aligned}
\int_{\underline{x}} \underline{F} \cdot d\underline{s} &= \int_{t=0}^{t=1} \left(\begin{bmatrix} t \\ 2t^2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4t \\ 3t^2 \end{bmatrix} \right) dt \\
&= \int_0^1 t + 8t^3 + 3t^2 dt \\
&= \frac{1}{2} + 2 + 1 = \frac{7}{2}
\end{aligned}$$

2. Let $\underline{F} = \begin{bmatrix} e^x \\ y^2 \end{bmatrix}$ and $\underline{x}(t) = \begin{bmatrix} t \\ \sin(t) \end{bmatrix}$, $t \in [0, \pi]$. Then, $\underline{x}'(t) = \begin{bmatrix} 1 \\ \cos(t) \end{bmatrix}$ and

$$\begin{aligned}
\int_{\underline{x}} \underline{F} \cdot d\underline{s} &= \int_{t=0}^{t=\pi} \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t) dt \\
&= \int_0^\pi \begin{bmatrix} e^t \\ \sin^2(t) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \cos(t) \end{bmatrix} dt \\
&= \int_0^\pi \left(e^t + \sin^2(t)\cos(t)\right) dt \\
&= \left[e^t + \frac{1}{3}\sin^3(t)\right]_0^\pi = e - 1
\end{aligned}$$

Remark: We can also define vector line integrals along piecewise C^1-paths $(\underline{x}_1, \dots, \underline{x}_k)$

$$\int_{(\underline{x}_1, \dots, \underline{x}_k)} \underline{F} \cdot d\underline{s} \stackrel{def}{=} \sum_{i=1}^{k} \int_{\underline{x}_i} \underline{F} \cdot d\underline{s}$$

Convention: For the rest of the course we will assume that all vector fields are continuous.

We are going to use vector line integrals to provide a solution to the Potential Function Problem.

Crucial Observation: Let $\underline{F} = \nabla f$ be a conservative vector field. Then, if $\underline{x}(t)$, $t \in [a, b]$, is a (piecewise) C^1-path then

$$\underline{F}(\underline{x}(t)) \cdot \underline{x}'(t) = \nabla f(\underline{x}(t)) \cdot \underline{x}'(t) = D(f \circ \underline{x})(t) = \frac{d}{dt}(f \circ \underline{x})(t)$$

Hence,

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s} = \int_a^b \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t) dt = \int_a^b \frac{d}{dt}(f \circ \underline{x}) dt = f(\underline{x}(a)) - f(\underline{x}(b)) = f(Q) - f(P)$$

where we write $P = \underline{x}(a)$, $Q = \underline{x}(b)$ for the endpoints of $\underline{x}$. If $\underline{y}$ is **any** other C^1-path from P to Q then we would also find (exercise!)

$$\int_{\underline{y}} \underline{F} \cdot d\underline{s} = f(Q) - f(P).$$

In particular,

Independence of path

Let $\underline{F}$ be a conservative vector field with domain $X \subset \mathbb{R}^2$. Let $P, Q \in X$ and suppose $\underline{x}, \underline{y}$ are C^1-paths in X both having start point P and end point Q. Then,

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s} = \int_{\underline{y}} \underline{F} \cdot d\underline{s}.$$

We say that a C^1-path $\underline{x}(t)$, $t \in [a,b]$, is **closed** if $\underline{x}(a) = \underline{x}(b)$. More generally, a piecewise C^1-path $(\underline{x}_1, \dots, \underline{x}_k)$ is **closed** if $\underline{x}_1(a_1) = \underline{x}_k(b_k)$ (using notation as in the Remark above).

We have the following consequence for vector line integrals of conservative vector fields along closed C^1-paths.

Vanishing along closed paths

Let $\underline{F}$ be a conservative vector field with domain $X \subseteq \mathbb{R}^2$. Let $\underline{x}$ be a closed C^1-path in X. Then,

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s} = 0$$

In particular,

Line Integral Test for non-Conservative Vector Fields

Let $\underline{F}$ be a vector field with domain $X \subseteq \mathbb{R}^2$. If there exists a closed C^1-path $\underline{x}(t)$ in X such that the vector line integral of $\underline{F}$ along $\underline{x}$ is nonzero then $\underline{F}$ is not conservative on X.

Remark: Analogous results hold for vector fields defined on $X \subset \mathbb{R}^n$.

Example: Consider the vector field

$$\underline{F} = \begin{bmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{bmatrix}$$

and consider the closed C^1-path

$$\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \quad t \in [0, 2\pi]$$

Then,

$$\begin{aligned} \int_{\underline{x}} \underline{F} \cdot d\underline{s} &= \int_{t=0}^{t=2\pi} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt \\ &= \int_0^{2\pi} dt = 2\pi \neq 0 \end{aligned}$$

Therefore, $\underline{F}$ is **not conservative**!.