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# April 11 Lecture

## Line Integrals

Learning Objectives:

- *Understand the definition of a scalar line integral.*
- *Understand the definition of a vector line integral.*
- *Learn how to compute basic line integrals.*

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In our last lecture we saw the following result:

**Local existence of potential functions**

Let $\underline{F} : X \subset \mathbb{R}^2 \to \mathbb{R}^2$ be vector field on $\mathbb{R}^2$, and write $\underline{F}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$.

Suppose that

- $X$ is either the whole plane, an open rectangle or an open disc,
- $u, v$ have continuous partial derivatives,
- $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$.

Then, there exists a differentiable function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = \underline{F}$.

Today we introduce an essential tool that we will use in order to understand how we can determine the Potential Function Problem for vector fields $\underline{F}$ on arbitrary domains $X$ - this is the notion of a **line integral**. We will see several types of line integral - **scalar line integrals** and **vector line integrals**. These integrals will require understanding a formula for **length of a path** $\underline{x}(t)$.

**Remark:** Line integrals are also called **path integrals**, **curve integrals**, or **contour integrals**.

### Length of differentiable paths

Let $\underline{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ be a differentiable path in $\mathbb{R}^n$ and assume that $\underline{x}'(t)$ is continuous. For $a, b \in I$, $a < b$, we define the **length of** $\underline{x}(t)$ **between** $t = a$ **and** $t = b$ to be

$$\int_{t=a}^{t=b} |\underline{x}'(t)| dt$$

**Remark:** Differentiable paths $\underline{x}(t)$ on $\mathbb{R}^n$ such that $\underline{x}'(t)$ is continuous will be called **$C^1$-paths**.

**Example:**

1. Let $\underline{x}(t) = \begin{bmatrix} 1+t \\ 2-t \\ 3t \end{bmatrix}$, $t \in \mathbb{R}$, be the straight line parallel to $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ passing through $(1,2,0)$. The length of $\underline{x}(t)$ between $t=0$ to $t=2$ is

$$\begin{aligned} \int_0^2 |\underline{x}'(t)|dt &= \int_0^2 \sqrt{1^2+(-1)^2+3^2}dt \\ &= \sqrt{11}\int_0^2 dt = 2\sqrt{11} \end{aligned}$$

You can check (exercise!) that this is the length of the line segment between $P = \underline{x}(0) = (1,2,0)$ and $Q = \underline{x}(2) = (3,0,6)$.

In general, if $\underline{x}(t) = P + t\underline{v}$ is a line with direction vector $\underline{v}$ then the length of $\underline{x}(t)$ from $t=a$ to $t=b$ is $(b-a)|\underline{v}|$. This is equal to the length of the line segment from $\underline{x}(a)$ to $\underline{x}(b)$.

2. Let $\underline{x}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$, $t \in I$, where $f(t)$ is a differentiable function whose derivative is continuous. Then, the image curve of $\underline{x}(t)$ is the graph of $f$. We compute the length of $\underline{x}$ between $t=a$ and $t=b$ to be

$$\int_a^b |\underline{x}'(t)|dt = \int_a^b \sqrt{1+(f'(t))^2}dt$$

This is the arc length formula from Calculus II.

3. Let $\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$, $t \in [0, 2\pi]$. The image curve of $\underline{x}(t)$ is the unit circle centred at the origin. Then, the length of $\underline{x}(t)$ between $t=0$ and $t=2\pi$ is

$$\int_0^{2\pi} |\underline{x}'(t)|dt = \int_0^{2\pi} \sqrt{(-\sin(t))^2+(\cos(t))^2}dt = \int_0^{2\pi} dt = 2\pi$$

**Observation:** in each of the examples above the arc length formula is computing the length of the image curve of $\underline{x}(t)$ between $\underline{x}(a)$ and $\underline{x}(b)$. In general,

**the length formula computes the length of the curve traced out by $\underline{x}(t)$ between $\underline{x}(a)$ and $\underline{x}(b)$.**

**Remark:** The length formula is derived as follows when $\underline{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ is a differentiable path in $\mathbb{R}^3$: consider a partition $a = t_0 < t_1 < \ldots < t_n = b$. Then, summing the lengths of the line segments $L_i$, $i = 1, \ldots, n$ between $\underline{x}(t_{i-1})$ and $\underline{x}(t_i)$ gives an approximation

$$s = \sum_{i=1}^n |\overrightarrow{\underline{x}(t_{i-1})\underline{x}(t_i)}| = \sum_{i=1}^n \sqrt{(x(t_i)-x(t_{i-1}))^2+(y(t_i)-y(t_{i-1}))^2+(z(t_i)-z(t_{i-1}))^2}$$

to the length of $\underline{x}(t)$ between $t = a$ and $t = b$.

Apply the Mean Value Theorem (three times) to find $a_i \in [x(t_{i-1}), x(t_i)]$, $b_i \in [y(t_{i-1}), y(t_i)]$, $c_i \in [z(t_{i-1}), z(t_i)]$ so that

$$x'(a_i) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}, \quad y'(b_i) = \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}}, \quad z'(c_i) = \frac{z(t_i) - z(t_{i-1})}{t_i - t_{i-1}}$$

Write $\Delta t_i = t_i - t_{i-1}$. Thus,

$$s = \sum_{i=1}^{n} \sqrt{(x'(a_i))^2 + (y'(b_i))^2 + (z'(c_i))^2} \Delta t_i$$

Then, the length of $\underline{x}(t)$ between $t = a$ and $t = b$ is

$$\lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{(x'(a_i))^2 + (y'(b_i))^2 + (z'(c_i))^2} \Delta t_i$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_a^b |\underline{x}'(t)| dt$$

A similar argument can be used for differentiable paths in $\mathbb{R}^n$.

## Scalar line integrals

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\underline{x}(t) : [a, b] \to \mathbb{R}^n$ a $C^1$-path.

### Scalar line integrals

The **scalar line integral of $f$ along $\underline{x}$** is

$$\int_a^b f(\underline{x}(t)) |\underline{x}'(t)| dt$$

We also write

$$\int_{\underline{x}} f ds$$

**Intepretation:** Suppose that $\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$, $t \in [a, b]$, is a $C^1$-path modelling a piece of metal coil. Let $f(x, y, z)$ be a density function for the coil. Then, $\int_{\underline{x}} f ds$ determines the total mass of the metal coil.

**Example:**

1. Let $\underline{x}(t) = \begin{bmatrix} 1+t \\ 2-t \\ 3t \end{bmatrix}$, $t \in [0,2]$, be the line segment from above, and let $f(x,y,z) = z$. Then,

$$\begin{aligned} \int_{\underline{x}} f ds &= \int_{t=0}^{t=2} f(\underline{x}(t))|\underline{x}'(t)|dt \\ &= \int_0^2 3t\sqrt{11}dt = 6\sqrt{11} \end{aligned}$$

2. Let $\underline{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$, $t \in [0,\pi]$, and let $f(x,y,z) = 2xy$. Then,

$$\begin{aligned} \int_{\underline{x}} f ds &= \int_{t=0}^{t=\pi} f(\underline{x}(t))|\underline{x}'(t)|dt \\ &= \int_0^\pi 2\sin(t)\cos(t)\sqrt{\cos^2(t)+\sin^2(t)+1}dt \\ &= \sqrt{2}\int_0^\pi \sin(2t)dt = \frac{1}{\sqrt{2}}\left[-\cos(2t)\right]_0^\pi = \frac{1}{\sqrt{2}}(1-1) = 0 \end{aligned}$$

## Vector line integrals

Now we see how to integrate a vector field along a path. Let $\underline{F}$ be a vector field on $\mathbb{R}^n$, $\underline{x}(t)$ a $C^1$-path in $\mathbb{R}^n$.

### Vector line integrals

The **vector line integral of $\underline{F}$ along $\underline{x}$** is

$$\int_{\underline{x}} \underline{F} \cdot d\underline{s} = \int_a^b \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t)dt$$

**Example:** Let $\underline{F} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ and consider the path $\underline{x}(t) = \begin{bmatrix} t \\ 2t^2 \\ t^3 \end{bmatrix}$, $t \in [0,1]$. We compute the vector line integral of $\underline{F}$ along $\underline{x}$ as follows:

$$\begin{aligned} \int_{\underline{x}} \underline{F} \cdot d\underline{s} &= \int_{t=0}^{t=1} \left( \begin{bmatrix} t \\ 2t^2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4t \\ 3t^2 \end{bmatrix} \right) dt \\ &= \int_0^1 t + 8t^3 + 3t^2 dt \\ &= \frac{1}{2} + 2 + 1 = \frac{7}{2} \end{aligned}$$