Mathematics Professors and Mathematics Majors' Expectations of Lectures in Advanced Mathematics

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Introduction

The advanced proof-oriented courses for mathematics majors are typically taught in a lecture format, where much of the lecture is comprised of presenting definitions, theorems, and proofs. There is a general perception amongst mathematicians and mathematics educators that these lectures are not as effective as they could be. However, the issues of why these lectures are not effective and how they might be improved are not discussed often in the mathematics education literature. In my research, I have sought to address this issue. Through task-based interviews with students and discussions about pedagogy with mathematicians, as well as observations of lectures and students' reactions to them, I have found that mathematics professors and mathematics majors have different expectations of lectures and these different expectations lead to barriers in communication. By expectations, I am referring to (i) what a student is supposed to learn from, or "get out of", a lecture and (ii) how students should engage in the lecture to understand this content. As a consequence of these different expectations, students do not gain what mathematicians hoped they would from the lectures they attend. Below I describe four such differing expectations and how they might inhibit lecture comprehension, with the hope that discussing these differing expectations might help us improve the teaching of proof at the undergraduate level.

Expectation #1 – Students can learn a lot by filling in the logical details of the presented proofs

In a conversation with a mathematician about his teaching, he provided the following anecdote:

"I'm doing a reading course with a student on wallpaper groups and there is a very elegant, short proof on the classification of wallpaper groups written by an English mathematician. [...] he's deliberately not drawing pictures because he wants the reader to draw pictures. And so I'm constantly writing in the margin, and trying to get the student to adopt the same pattern. Each assertion in the proof basically requires writing in the margin, or doing an extra verification, especially when an assertion is made that is not so obviously a direct consequence of a previous assertion [...] I write sub-proofs and check lots of examples" (Weber & Mejia-Ramos, 2011, p. 337).

might consist of drawing pictures, considering examples, and writing sub-proofs. This is not only about checking the correctness of a proof; it is also an important means of building one's understanding of mathematics. Indeed, mathematics professors feel that we may be robbing students of this opportunity if we did all of this work for them (cf., Lai, Weber, & Mejia-Ramos, 2012) and research has shown that encouraging students to meaningfully connect new statements within a proof with previous statements and their own knowledge base enhances proof comprehension (Hodds, Inglis, & Alcock, 2014).

Students see things differently. In an interview study, I asked 28 mathematics majors what makes a good mathematical proof. The following response was typical.

"For me, as a student, what else I would like to see is all the intermediate sorts of steps, things to help along, graphs, visual things. Things that recalled facts that perhaps I should know but you know, maybe not immediately at the tip of my tongue. That's to me what makes a good mathematical argument" (Weber & Mejia-Ramos, 2014).

Sixteen of the 28 students gave a response like this, claiming all the intermediate steps of a proof should be provided. If students have to work at understanding the proof, then from their perspective, this was not a good proof. A survey with 175 upper-level mathematics majors and 83 mathematics professors verified the generality of these findings. Most mathematics majors (75%) agreed with a statement that a student should not have to spend time filling in logical gaps if a good proof was presented with them, while few mathematicians (27%) agreed with this statement (Weber & Mejia-Ramos, 2014).

It is easy to see how these different expectations can hinder communication in mathematics lectures. For a variety of reasons, mathematics professors leave some details out of the proofs that they present. Typically, mathematics majors will not view filling in these gaps as their responsibility or an opportunity to learn; rather they will simply see the incomplete proof as a low quality presentation by the lecturer.

Expectation #2 – Understanding a proof is more than just providing a justification for each individual step

While a complete understanding of a proof includes understanding how each step in the proof was justified, most mathematicians believe understanding a proof involves more than this. For instance, some proofs might demonstrate a technique that can be useful for solving other problems or proving other theorems, others might illustrate a new way of thinking about a concept (Mejia-Ramos & Weber, 2014; Weber, 2010a). When presenting some proofs to students, they hope that students learn these things (Weber, 2012). Most mathematics majors disagree with this. The large majority (75%) believe that if they understand how every step in a proof follows from previous work, they understand the proof completely (Weber & Mejia-Ramos, 2014). Consequently, mathematics majors tend not to invest the time trying to understand proofs holistically, in part because they are not aware that this is something they should do.

Expectation #3 – Students are expected to spend time studying proofs outside of class

Most mathematics professors believe that understanding a proof can be a lengthy and complicated process (Weber, 2012). When asked how long a mathematics major should ideally spend studying a proof outside of the classroom, their average response was between 30 and 37 minutes, with 81% giving a response of greater than 15 minutes. Again, mathematics majors see things differently. Their average response to the same question was 17 to 20 minutes, with only 41% giving a response of greater than 15 minutes. (Weber & Mejia-Ramos, 2014). In practice, mathematics majors probably spend much less time than that. In a study that they often did not fully understand the proofs that they were reading (Weber, 2010b).

Expectation #4 – What the professor is saying when he or she writes the proof is important

The job of a mathematics lecturer in advanced mathematics is hard. Two responsibilities of the lecturer are to help students (i) understand what constitutes a proof in advanced mathematics (i.e., what is an acceptable product) and (ii) understand how proofs can be written (i.e., what is the process by which that product is produced). These two things are sometimes in opposition to one another. Writing proofs effectively involves knowing how to choose a proving approach that is likely to be effective (e.g., Weber, 2001), but the proof itself focuses on the implementation of the approach. The proof itself is a verification, not a description of the problem-solving process (Selden & Selden, 2013).

One way that lecturers manage these competing responsibilities is by leaving a space on the blackboard designated for the "official proof", while discussing how the proof was produced orally (Fukawa-Connelly et al, 2014; Weber, 2004). What the professor says aloud but does not write down is therefore a crucial part of his or her lecture.

We studied a lecture where a professor repeatedly emphasized the same important proving heuristic in a proof that he provided. After the lecture, we asked six students what they learned from this proof and none mentioned this heuristic. When we showed each student a video-recording of this part of the lecture again, we obtained the same result. Inspecting the notes that the students took offered an explanation for this phenomenon. Only one student wrote down anything the professor said aloud. The other five students' notes were comprised entirely of what the professor wrote on the board (Fukawa-Connelly et al, 2014). Clearly if students focus on what the professor writes but not what he or she says, much of the insight in the professor's oral comments will not be acknowledged by the student.

Discussion and recommendations

I have outlined four expectations that students hold about mathematics lectures which are at variance with mathematicians' intentions and inhibit what they learn from these lectures. A point that I wish to emphasize is that I do not believe it is the students' fault that they have these expectations. Lectures in advanced mathematics are a new experience for students. They need guidance about the nature of these lectures and what the role of this process should be. (*How to study for a mathematics degree* (http://ukcatalogue.oup.com/product/9780199661329.do) (Alcock, 2012) is a useful resource for students in this regard). Indeed, some lecturers hold the belief that the purpose of the lecture is to provide the students with the big ideas, intuitions, and motivations for a proof rather than focus on its logical details (Lai & Weber, 2014), but students do not view lectures in this way. Further, many of the students' expectations are quite sensible given their previous experiences. For instance, in students' previous exposure to proof, they were probably required to justify every claim that they made, even ones that seemed to be obvious. This is certainly the case– indeed, the motivation– for the two-column proofs in high school geometry. It is therefore understandable that they would expect their professors to provide a similar level of justification. We should not expect that students will naturally share mathematicians' expectations; the professors will have to work to develop this shared understanding. Below are some recommendations as to how this can be done.

Explicitly communicate your expectations about lectures early in the semester. One reason for the discrepancy between mathematics professors and students' expectations of lectures in mathematics is that these expectations are rarely the topic of explicit conversation. Stating what you hope to convey in your

Assess what you want your students to know.
I have found that advanced mathematics lecturers strive to engage students with high-level or intuitive ways to understand the course content, but students' assessments are predominantly on formal mathematics (i.e., stating definitions, writing proofs). This sends a message to students that the formal aspects of mathematics are what is important and the other aspects of mathematics are superfluous. I would recommend assessing students on high-level and informal aspects of mathematical proofs as well. Of course, this will motivate students to understand the material, but there are two additional benefits. First, the types of questions that students are asked can give them a better sense of how the material should be understood. Second, students' responses to these questions will provide the instructor with a better sense of how the students are interpreting his or her lecture.

My colleagues and I have developed a template for asking questions designed to measure students' understanding of a proof (Mejia-Ramos et al, 2012). These include questions regarding the holistic nature of a proof, such as asking students to apply the ideas of a proof to a specific example or diagram, having students use the technique in the proof to establish a different theorem, or presenting students with several summaries and asking which one captures the main idea of a proof that they read. These questions illustrate to students that understanding a proof does not solely consist of justifying each step within a proof and it provides incentive to study lecture proofs carefully outside of class.

Write down your key points. A typical lecturer speaks at a rate of over 100 words a minute while a typical student can transcribe less than 25 words a minute. Students cannot write down everything they hear; they must prioritize. Further, nearly everything that students do not write down is not recalled at a later time (Williams & Eggert, 2002). It is natural for students to focus on what is written on the blackboard. Written statements have a permanence that oral speech lacks and it is a traditional means by which a professor emphasizes importance. I would recommend that the key ideas that a professor wishes to emphasize in a proof should be written down on the blackboard or distribute to the students as lecture notes. If they are not, students will probably not include them in their notes and will consequently forget these lessons.

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