Math 122C: Series &
Multivariable Calculus
Fall 2018

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September 7 Summary

Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.1

Keywords: *sequences, increasing/decreasing, bounded above/below, bounded, convergent, limit, Squeeze Theorem*

Sequences

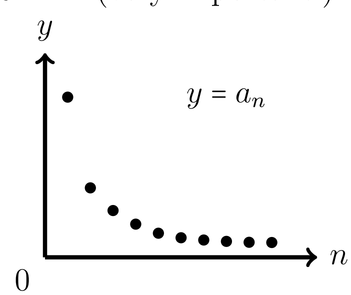
• A **sequence** is an infinite list of real numbers; a_n is the n^{th} **term**. Denoted $\{a_n\}_{n=1}^{\infty}$ or, simply, $\{a_n\}$.
• Sequences can be defined using **formulae** or **in words**:

1. $a_n = n^2$: $1, 4, 9, 16, 25, \ldots$
2. $a_n = \cos(\pi/n)$: $-1, 0, 1/2, 1/\sqrt{2}, 0.809..., \sqrt{3}2, \ldots$
3. $a_n = (-1)^n$: $-1, 1, -1, 1, -1, 1, -1, \ldots$
4. $a_n = n^{th}$ prime number: $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$
5. $a_n = n^{th}$ digit of π: $1, 4, 1, 5, 9, 2, 6, 5, \ldots$

Representing sequences: **lists** or **graphs** e.g. $\{a_n\}_{n=1}^{\infty}$, $a_n = 1/n$

LIST: $1, 1/2, 1/3, 1/4, \ldots$

GRAPH (very important!):



This is the graph of a function $f(n) = a_n$, having inputs n being positive integers

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Describing sequences
Say a sequence $\{a_n\}$ is:

• **bounded above** if there is a real number M (called an **upper bound**) satisfying

$$a_n \le M, \qquad \text{for every } n = 1, 2, 3, \ldots$$

i.e. $a_1 \le M$ and $a_2 \le M$ and $a_3 \le M$ etc.

• **bounded below** if there is a real number m (called a **lower bound**) satisfying

$$m \le a_n, \qquad \text{for every } n = 1, 2, 3, \ldots$$

i.e. $m \le a_1$ and $m \le a_2$ and $m \le a_3$ etc.

• **bounded** if both bounded above and bounded below; **unbounded** if not bounded.

• **nondecreasing** if $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le \ldots$

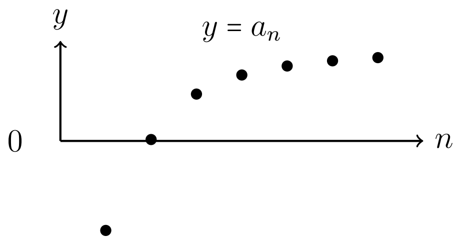
• **nonincreasing** if $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge \ldots$

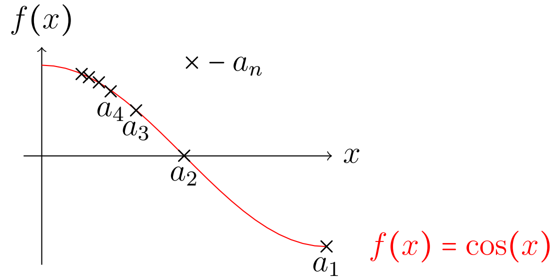
• **monotone** if either nonincreasing or nondecreasing.

Examples (from above):

1. Bounded below (by $m = 0$, say) nondecreasing, **not** bounded above

2. Bounded ($-1 \le \cos(\pi/n) \le 1$, for all $n = 1, 2, 3, \ldots$), nondecreasing





3. Bounded ($-1 \le a_n \le 1$, for every $n = 1, 2, 3, \ldots$), not monotone (neither nondecreasing nor nonincreasing)

4. Nondecreasing, bounded below, not bounded above

5. Bounded ($0 \le a_n \le 9$, for every $n = 1, 2, 3 \ldots$), not monotone.

Behaviour 'at infinity'

Q: What's the limiting behaviour of $\{a_n\}$ when we let n get very large? i.e. what happens far to the right in the graph?

• Say $\{a_n\}$ converges to the real number L if $y = L$ is a horizontal asymptote of the graph. In this case, write $L = \lim_{n\to\infty} a_n$ and say $\{a_n\}$ **converges to** L or, simply, $\{a_n\}$ is **convergent**. If the graph of $\{a_n\}$ does not possess a horizontal asymptote then say $\{a_n\}$ diverges.

• **Remark:** the rigorous mathematical definition of convergence will be investigated in Homework 1.

Important Examples:

a) $\lim_{n\to\infty} \frac{1}{n} = 0$,

b) Let x be a real number. Then,

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & -1 < x \le 1 \\ 1, & x = 1 \end{cases}$$

If $x \le -1$ or $x > 1$ then $1, x, x^2, x^3, \ldots$ is divergent.

• **Important:** divergent = not convergent

• If $\{a_n\}$ increases/decreases without bound then $\{a_n\}$ is **divergent to** $\pm\infty$. Write (confusingly) $\lim_{n\to\infty} a_n = \pm\infty$.

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1. Diverges to $+\infty$, $\lim_{n\to\infty} n^2 = +\infty$
2. Converges to $L = 1$. **Note:** there's no n so that $\cos(\pi/n) = 1$; but the sequence approaches $L = 1$ as n gets very large.
3. Divergent.
4. Diverges to $+\infty$ (this relies on the following Theorem (originally due to Euclid): there are an infinite number of primes)
5. Diverges: this is because π is irrational (though this fact is non-obvious, neither is why this implies the sequence is divergent).

Limit Laws: Let $\{a_n\}$, $\{b_n\}$ be convergent sequences. Let c be a constant.

- $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ i.e. *the sum of convergent sequences is convergent*
- $\lim_{n\to\infty} ca_n = c(\lim_{n\to\infty} a_n)$
- $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$ i.e. *product of convergent sequences is convergent*
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^r = (\lim_{n\to\infty} a_n)^r$, provided $a_n \geq 0$

Determining limits/convergence: (HARD, in general)

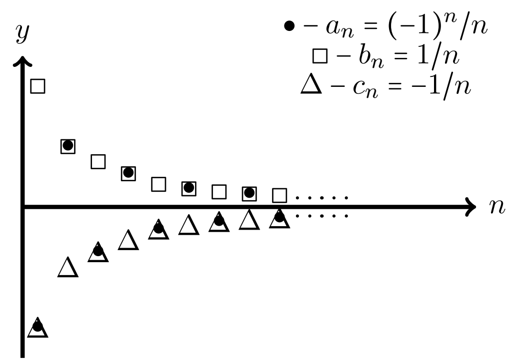
Squeeze Theorem
Let $\{a_n\}$ be a sequence. Suppose there are convergent sequences $\{b_n\}$ and $\{c_n\}$ satisfying

- $c_n \leq a_n \leq b_n$, for every $n = 1, 2, 3, \ldots$, and
- $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = L$

Then, $\{a_n\}$ is convergent and $L = \lim_{n\to\infty} a_n$.

Example:

1. Let $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{n}$, $c_n = -\frac{1}{n}$. Then, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = 0$ and $c_n \leq a_n \leq b_n$. Hence, $\{a_n\}$ is convergent, by the Squeeze Theorem, and $\lim_{n\to\infty} a_n = 0$.



2. Let $a_n = \sin(n)/n$. Let $b_n = 1/n$, $c_n = -1/n$. Then, $c_n \leq a_n \leq b_n$ (since $-1 \leq \sin(n) \leq 1$) and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = 0$. Hence, $\{a_n\}$ is convergent, by the Squeeze Theorem, and $\lim_{n\to\infty} a_n = 0$.