Math 122C: Series &
Multivariable Calculus
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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.5

Keywords: *power series*

Power Series

• **Recall:** a **power series** is a series of the form

$$_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

where x is a variable; c is the **centre**; $a_0, a_1, a_2, \dots$ are **coefficients**.

• **Basic question:** For which series x does the power series converge?
• **Remark:** If I is the collection of all x for which a power series converges then we can define a function

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots, \quad x \text{ in } I$$

Example:

1. Any power series always converges at $x = c$; if $x = c$ then the series becomes

$$a_0 + 0 + 0 + 0 + \dots = a_0$$

2. Consider the power series $\sum_{k=0}^{\infty} k! x^k$. Since

$$\left| \frac{(k+1)! x^{k+1}}{k! x^k} \right| = (k+1)|x|$$

the series does not converge, **for any** $x \neq 0$.

3. $\sum_{k=1}^{\infty} \frac{(x+2)^k}{k}$: let $c_k = \frac{(x+2)^k}{k}$. Then

$$\left| \frac{c_{k+1}}{c_k} \right| = \frac{|x+2|k}{(k+1)} \to |x+2|, \quad \text{as } k \to \infty$$

   - converges (absolutely) if $|x+2| < 1$ i.e. $-3 < x < -1$
   - diverges if $|x+2| > 1$ i.e. $x < -3$ or $x > -1$.
   - $x = -1$: the series becomes $\sum \frac{1}{k}$, divergent;
     $x = -3$: the series becomes $\sum (-1)^k \frac{1}{k}$, which is convergent, by AST.

   Hence, the series converges when $-3 \leq x < -1$.

4. Similarly, $\sum_{k=1}^{\infty} \frac{(x+2)^k}{k^2}$ converges when $-3 \leq x \leq -1$.

Given a power series, one of three situations occur:

- power series converges only at $x = c$;
- power series converges on a finite interval;
- series converges for all x.

We call this interval the **interval of convergence of the power series**.