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Supplementary References:
- *Multivariable Calculus...*, Ostebee-Zorn, Section 11.5

Keywords: *power series*

Power Series

- **Recall:** the series $\sum_{k=0}^{\infty} x^k$ converges when $-1 < x < 1$ and $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

In particular, the function $f(x) = \frac{1}{1-x}$ can be represented as a series whenever $-1 < x < 1$.

- We can extend this idea: define $g(x) = \frac{1}{5-x}$, defined when $x \neq 5$. Note

$$\frac{1}{5-x} = \frac{1}{5} \cdot \frac{1}{1-x/5} \stackrel{*}{=} \frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{x}{5}\right)^k = \frac{1}{5}\left(1 + \left(\frac{x}{5}\right) + \left(\frac{x}{5}\right)^2 + \dots\right)$$

$$= \frac{1}{5} + \frac{x}{25} + \frac{x^2}{125} + \frac{x^3}{625} + \dots$$

The equality $*$ holds whenever $-1 < \frac{x}{5} < 1$ i.e. $-5 < x < 5$. That is, we've given a **series representation for $g(x)$, valid whenever** $-5 < x < 5$.

- A **power series** is a series of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

Here: x is a variable; $a_0, a_1, a_2, \dots$ are **coefficients**; c is the **centre/base point**.

Example

1. The power series

   $$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

   has centre $c = 0$; coefficients $a_k = \frac{1}{k!}$.

   Question: for which x does the series converge?

   We use the following :

> **Extension of the Ratio Test**
>
> Let $\sum a_k$ be a series. Denote $L = \lim \left|\frac{a_{k+1}}{a_k}\right|$.
>
> - If $L < 1$ then the series converges absolutely.
> - If $L > 1$ then the series diverges.
> - If $L = 1$ then further testing is required.

Let $a_k = \frac{x^k}{k!}$. Then,

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{(k+1)!}\frac{k!}{x^k}\right| = \frac{|x|}{(k+1)} \to 0, \quad \text{as } k \to \infty$$

Hence, the series converges for all x.

2. Consider the power series

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{3^k} = 1 + \frac{(x-1)}{3} + \frac{(x-1)^2}{9} + \dots$$

The centre is $c = 1$; coefficients $a_k = \frac{1}{3^k}$. Again, we want to determine those x for which the series converges. We proceed as above.

Let $a_k = \frac{(x-1)^k}{3^k}$. Then,

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x-1|}{3} \to \frac{|x-1|}{3}, \quad \text{as } k \to \infty$$

Hence

- The series converges (absolutely) when $\frac{|x-1|}{3} < 1$ i.e. $|x-1| < 3$ i.e. $-2 < x < 4$
- The series diverges when $|x-1| > 3$ i.e. $x < -2$ or $x > 4$.
- We have to check what happens when $|x-1| = 3$ i.e. $x = -2$ or $x = 4$.
  - $x = -2$: the series becomes $\sum_{k=0}^{\infty} \frac{(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k$, which is divergent.
  - $x = 4$: the series becomes $\sum_{k=0}^{\infty} \frac{3^k}{3^k} = \sum_{k=0}^{\infty} 1$, which is divergent. Hence, the power series converges for $-2 < x < 4$.